TORSION PENDULUM

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ABSTRACT: This experiment used a torsion pendulum apparatus to calculate the damping constant, β, at points where the system is under dampened and critically dampened. This experiment also used a forcing motor to drive the pendulum at frequencies near resonance for three different brake currents and used the calculated resonance frequencies to determine β for each case. In this experiment, we determined that β is not constant for free rotational oscillations but is for forced rotational oscillations.

INTRODUCTION:

This experiment examines the behavior of a torsion pendulum, a special case among mechanical oscillators.

Note that angular frequency (w in rad/s) and frequency (f in Hz.) are not the same.
In the damped case, the torque balance for the torsion pendulum yields the differential equation:

$$J\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + c\theta = 0 \tag{1}$$

where J is the moment of inertia of the pendulum, b is the damping coefficient, c is the restoring torque constant, and θ is the angle of rotation [*Leybold Scientific,*2006a]. This equation can be rewritten in the standard form [*Thorton and Marion,* 2004]:

$$\ddot{\theta} + 2\beta\dot{\theta} + \omega_0^2\theta = 0, \tag{2}$$

where the damping constant is $\beta = \frac{b}{2J}$ and the natural frequency is $\omega_0 = \sqrt{\frac{c}{J}}$. The general solution to this differential equations is:

$$\theta(t) = e^{-\beta t}\left[A_1 e^{\sqrt{\beta^2-\omega_0^2}\,t} + A_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right], \tag{3}$$

with three different types of solutions possible depending on the relationships between ω_0 and β. In the underdamped case ($\beta < \omega_0$):

$$\theta(t) = \theta_0 e^{-\beta t}\cos(\omega_1 t - \gamma) \tag{4}$$

with the oscillation frequency $\omega_1 = \sqrt{\omega_0{}^2 - \beta^2}$, initial amplitude θ_0, and phase γ. In the critically damped case ($\beta = \omega_0$):

$$\theta(t) = (A + Bt)e^{-\beta t}. \tag{5}$$

In the overdamped case ($\beta > \omega 0$):

$$\theta(t) = e^{-\beta t}[A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t}], \tag{6}$$

where $\omega_2 = \sqrt{\beta^2 - \omega_0^2}$.

For the forced oscillation case, an external torque is added to Equation 1:

$$J\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + c\theta = \tau_0 \sin(wt), \tag{7}$$

where ω is the driving frequency and τ_0 is the driving torque [*Leybold Scientific,* 2006b]. The general solution to the differential equation is the sum of the homogeneous solutions (which are the solutions to the damped case above) plus a particular solution. The particular solution has the form:

$$\theta(t) = \theta_m(w)\sin(wt - \phi) \tag{8}$$

with

$$\theta_m(\omega) = \frac{\tau_0}{J\sqrt{(\omega_0 - \omega)^2 + \left(\frac{b\omega}{J}\right)^2}}. \tag{9}$$

In this case the resonance frequency is $\omega_r = \sqrt{\omega_0^2 - 2\beta^2}$ and the phase shift between the pendulum and the external oscillator is:

$$\tan\phi = \frac{2\beta\omega}{\omega_0^2 - \omega^2} \tag{10}$$

THE EXPERIMENT:

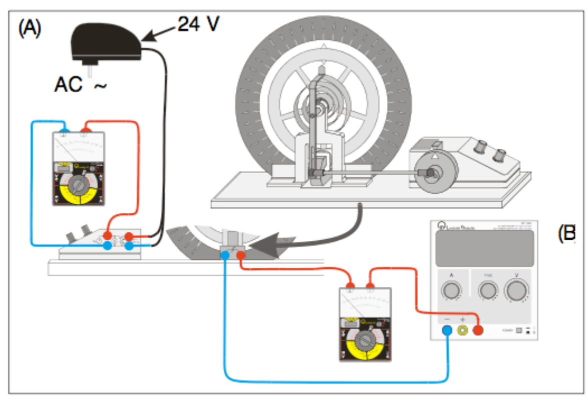

Figure 1: A schematic of the torsion pendulum apparatus [*Leybold Scientific, 2006b*]

The experiment was conducted with the torsion pendulum apparatus set up as shown in Figure 1, using a low voltage power supply and two digital multimeters.

For the first half of the experiment, the driving motor was not active.

We first determined was the period of the torsion pendulum to find the natural frequency by taking 3 measurements of 10 oscillations of the pendulum with the damping magnet turned off.

Next, we determined the damping constant, β, at two values of the damping current, 0.1 A < I_1 < 0.3 A and 0.3 A < I_2 < 0.6 A. First, we calculated the period of oscillation at each current and then started the pendulum at its furthest point of rotation and measured the amplitude after each period. We then fit the amplitude vs. time data to an exponential curve and used equation (5) to determine the damping constant.

The damping current was increased until the pendulum was near critically dampened, or when it only completed one oscillation after release from its furthest rotation point. For this case, we took the average oscillation time. We then further increased the current to the point where the pendulum was critically dampened, or did not cross zero after release from its furthest point of rotation. We used the critically dampened case to estimate the damping constant from equation (5) by measuring the time it took the pendulum to reach 1/10 of its initial value after being released from its furthest rotational point. In the critically dampened case, $B = 0$, A = furthest rotational point value at time t = 0. We used this β to get estimates of the damping coefficient, b, and restoring torque, c, where the moment of inertia of the pendulum.

The latter part of this experiment was conducted with the forcing motor turned on.

To determine a relationship between voltage and frequency of the forcing motor we determined the frequency of the motor at various voltages by finding the average period of the motor at several voltages and making a fit of frequency to obtain the relation between them.

The resonance frequency curve for currents of, I = 0A, I ~ 0A, I ~ 0.4A, I ~ 0.8A were determined by taking measurements of amplitude of oscillations in intervals of 0.005 Hz from $f = 0.465$ Hz to $f = 0.560$ Hz for each damping current by varying the voltage of the motor accordingly.

DATA PRESENTATION AND INTREPRETATION OF RESULTS:

Determination of the natural angular frequency, ω_0, with the damping current at I = 0 A :

Mean period for 10 oscillations	1.927 ± 0.005 s
Natural frequency	0.519 ± 0.0012 Hz
Natural angular frequency, ω_0	3.260 ± 0.008 rad/s

Table 1: the period, natural frequency and natural angular frequency of the torsion pendulum

Mean period at damping current I = 149 ± 3mA: 1.927 ± 0.005 seconds

The data for amplitude vs. time for a damping current of I = 149 ± 3mA was taken at each consecutive period starting after the pendulum had gone through one oscillation. The uncertainty in measurement of amplitude was assumed to be 0.2 at each measurement with an uncertainty in time given by: $\delta t = |n| * \delta T$. When this data was fit to an exponential curve, the fit equation was: $y = Ae^{Bx}$, where A = 21.2 ± 0.5 and B = -0.040 ± 0.001 rad/s. β was found to be -B as per equation (4): $\beta = 0.040 \pm 0.001$ rad/s.

For a damping current of 490 ± 7 mA, the mean period was found to be 1.927 ± 0.007 seconds

The data of amplitude vs. per consecutive period with a damping current of 490 ± 7 mA was taken in a similar fashion. Again, the uncertainty in amplitude was taken to be 0.2 and uncertainty in time was : $\delta t = |n| * \delta T$. This data was also fit to an exponential curve as seen in Figure 2 with a reduced chi-squared of 4.7. The equation of the fit line was of the form $y = Ae^{Bx}$ where A = 26.6 ± 0.4 and B = -0.17 ± 0.03 rad/s which, as per equation (4) gives the value of the damping constant to be $\beta = 0.17 \pm 0.03$ rad/s, a reasonable answer as one would expect the damping constant to increase as the damping current increases.

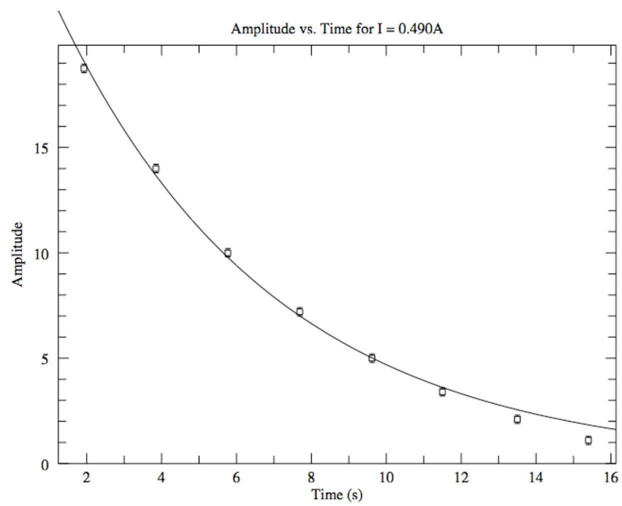

Figure 2: Plot of Amplitude vs. time fit to an exponential curve, where each data point is spaced apart by one period (1.927±0.007 seconds)

The system was near critically dampened at a damping current of 1.69 ± 0.02A with a mean stopping time of 2.3 ± 0.1 seconds. We found that the system was critically dampened at a damping current of 1.90 ± 0.02A with a mean stopping time of 1.6 ± 0.3 seconds. The results for the time to stop agree with the idea that the pendulum comes to rest at sooner when the damping is increased

For the critically dampened case, the mean time for θ to decrease from its furthest point of rotation $\theta_0 = 19$ to $\theta = 1.9$ was found to be 1.00 ± 0.08 seconds. We used this in conjunction with equation (5) to get estimates of β, b, and c. The damping constant was found by use of the following equation: $\beta = \frac{-\ln(\frac{\theta}{\theta_0})}{\Delta t}$ and uncertainty in β was found by $\delta\beta = \sqrt{(\frac{\partial\beta}{\partial\theta}\delta\theta)^2 + (\frac{\partial\beta}{\partial\theta_0}\delta\theta_0)^2 + (\frac{\partial\beta}{\partial\Delta t}\delta\Delta t)^2}$, where Δt = the mean time for θ to decrease to θ_0. The estimate of the damping coefficient, b was found by use of the following equation: $b = 2J\beta$ and $\delta b = \sqrt{(\frac{\partial b}{\partial J}\delta J)^2 + (\frac{\partial b}{\partial\beta}\delta\beta)^2}$, Where $J = 3.0 \pm 0.1$ kg • m^2. An estimation of the restoring torque was found by use of the following equation: $c = J\beta^2$ and $\delta c = \sqrt{(\frac{\partial c}{\partial\beta}\delta\beta)^2 + (\frac{\partial c}{\partial J}\delta J)^2}$. See Table 2 for the obtained values. In the critically dampened case, it is expected that $\beta = \omega_0$, which is not exactly the case as seen in comparing β from Table 2 and ω_0 from Table 1.

β = 2.3 ± 0.2 rad/s
b = 14 ± 1 (kg * m^2)/s
c = 16 ±3 kg^{-2}m^{-4}s^{-2}

Table 2: estimations of β, b, and c at critically dampened, I = 1.90±0.02A.

The latter part of this experiment dealt with forced oscillations of the pendulum.

A relationship between frequency and voltage was established by determining the mean period and thus frequency at a range of dial settings along with the voltage across the motor measured by the multimeter and fitting that data to a linear function.

The equation relating frequency and voltage is of linear of the form $y = A + Bx$, where A = -0.01 ± 0.006Hz, B = 0.071 ± 0.11x10^{-2} Hz/V, y is the frequency and x is the voltage across the motor.

We then chose the range of frequencies to look at resonance to be 0.465 Hz ≤ f ≤ 0.560 Hz for each damping current.

Measurements of amplitude vs. frequency were taken by setting the forcing motor to the particular voltage to give the desired frequency, stopping the pendulum before each measurement and letting the pendulum settle for a few minutes for each measurement.

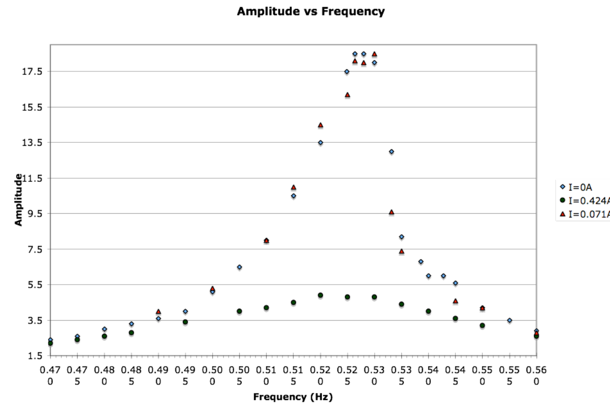

Figure 3: Plot of the resonance curves for damping currents of I = 0A, I = 72 ± 2mA, I = 424 ± 6mA

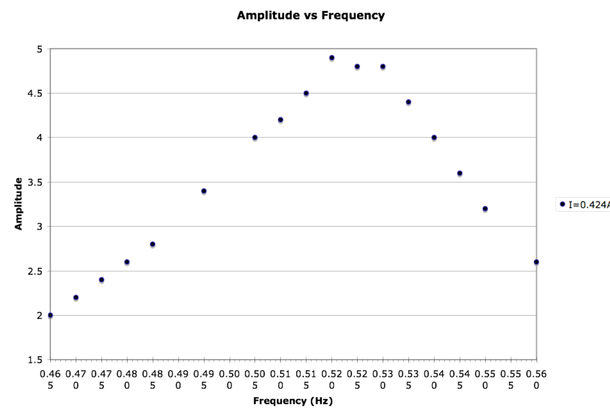

Figure 4: Plot of the resonance curve for a damping current of I=424±6mA

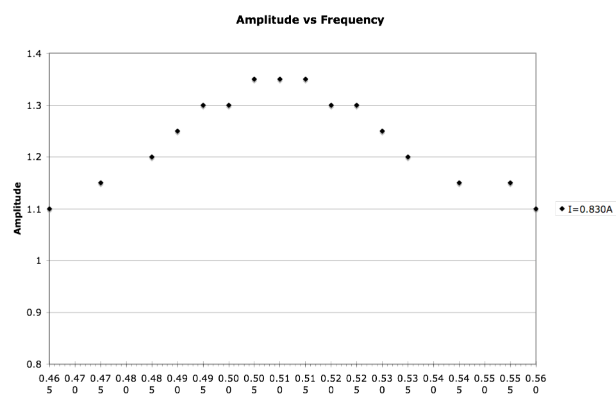

Figure 5: Plot of the resonance curve for a damping current of I = 0.830±0.01A

Figures 3 to 5 plot the resonance curves for braking currents I = 0A, I = 72 ± 2mA, I = 424 ± 6mA, and I = 0.83 ± 0.01A. The plots were used to determine the resonance frequency. The uncertainties in resonance frequency were estimated based on accuracy to which we could determine the frequency of resonance for each braking current. The damping constant was estimated by use of the following formula: $\beta = \sqrt{\frac{\omega_r - \omega_0}{2}}$, and the uncertainty of β was given by $\delta\beta = \sqrt{(\frac{\partial\beta}{\partial\omega_r}\delta\omega_r)^2 + (\frac{\partial\beta}{\partial\omega_0}\delta\omega_r)^2}$, with $\omega_0 = 3.26 \pm 0.02$. The values of f_r , ω_r, and β obtained are shown in Table 3 below.

	I=0A	I=71±2mA	I=424±7mA	I=830±10mA
f_r (Hz)	0.528 ±0.003	0.527 ± 0.003	0.523 ± 0.004	0.510 ± 0.007
ω_r (rad/s)	3.34 ±0.02	3.31 ± 0.02	3.29 ± 0.03	3.20 ± 0.04
β (rad/s)	0.51 ± 0.07	0.41 ± 0.09	0.3 ± 0.1	0.4 ±0.1

Table 3: resonance frequencies and the associated angular resonance frequencies, in addition to the calculated damping constant at each brake current.

As expected, the resonance frequency differs less from the natural frequency with lower damping.

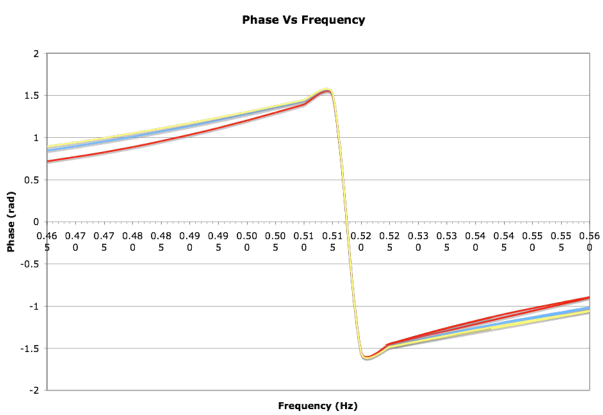

Figure 6: A plot of phase vs. frequency around the natural frequency of 0.519 ± 0.001 Hz

It can be seen in Figure 6 that as the difference between the angular driving frequency and the natural angular frequency approaches zero, the phase difference between the pendulum and the forcing motor approaches $\pm\frac{\pi}{2}$, and at high and low frequencies, the phase approaches zero. The phase plotted in Figure 6 was given by the equation: $\phi = \tan^{-1}(\frac{2\beta\omega}{\omega_0^2 - \omega^2})$, where ω is the angular frequency of the forcing motor, and ω_0 is the natural angular frequency.

	I=149±3mA	I=490±7mA	I=1.90±0.02A
β rad/s	0.040±0.001	0.17±0.03	2.3±0.2

Table 4: A comparison of β calculated for dampened oscillations at different damping currents

	I=0A	I=72±2mA	I=424±6mA	I=0.83±0.01A
β rad/s	0.51±0.07	0.41 ± 0.09	0.3 ± 0.1	0.4 ±0.1

Table 5: A comparison of β calculated using the resonance curve of forced oscillations at different braking currents

Comparing the damping constants found for dampened and forced oscillation at similar values of damping current, as shown in Tables 4 and 5, it is evident that β is not consistent. This could be a result of the poor fit for the damping current of I = 149±3mA. The damping constant for damping currents of ~0.45A are marginally close but do not agree within the bounds of uncertainty, which may again be a result of inaccuracies of the fit of the data.

REFRENCES:

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