Uniform Acceleration

We seek the solution to the problem of constant proper acceleration, a_0, where $\mathbb{A}^2 = a_0^2$. It will be convenient to define the parameter $\alpha = a_0/c$, which has the units of inverse time. In the case of motion along the x-axis, the 4-vectors reduce to two component vectors:

$$\begin{aligned} \mathbb{X} &= (x, ict) \\ \mathbb{U} &= \gamma(u, ic) \\ \mathbb{A} &= \gamma\left[\gamma(a,0) + \dot{\gamma}(u, ic)\right] \end{aligned}$$

where:

$$\dot{\gamma} = \frac{d\gamma}{dt} = \gamma^3 \, \frac{ua}{c^2}$$

So

$$\begin{aligned} \mathbb{A}^2 &= \gamma^2\left[(\gamma a + \dot{\gamma} u)^2 - (\dot{\gamma} c)^2\right] \\ &= \gamma^2\left[(\gamma a + \gamma^3\beta^2 a)^2 - (\gamma^3\beta a)^2\right] \\ &= \gamma^8\left[a^2\left(\frac{1}{\gamma^2} + \beta^2\right)^2 - a^2\beta^2\right] \\ &= \gamma^6 a^2 \\ &= a_0^2 \end{aligned}$$

So:

$$a_0 = \gamma^3 a = \frac{d}{dt}\,(\gamma u)$$

or

$$\alpha = \frac{d}{dt}\,(\gamma\beta) \quad \Rightarrow \quad \alpha t = \gamma\beta$$

where we've used:

$$\frac{d}{dt}\,\gamma u = \gamma^3\beta^2 a + \gamma a = \gamma^3 a(\beta^2 + 1/\gamma^2) = \gamma^3 a$$

A bit of algebra gives us separate expressions for β and γ as functions of time:

$$\begin{aligned} \gamma\beta &= \alpha t \qquad \text{we take } \beta = 0 \text{ at } t = 0 \\ (\gamma\beta)^2 = \frac{\beta^2}{1-\beta^2} &= (\alpha t)^2 \\ \beta^2(1 + (\alpha t)^2) &= (\alpha t)^2 \\ \beta &= \frac{\alpha t}{\sqrt{1 + (\alpha t)^2}} \\ \gamma &= \sqrt{1 + (\alpha t)^2} \end{aligned}$$

We can integrate $\beta(t)$ to find $x(t)$:

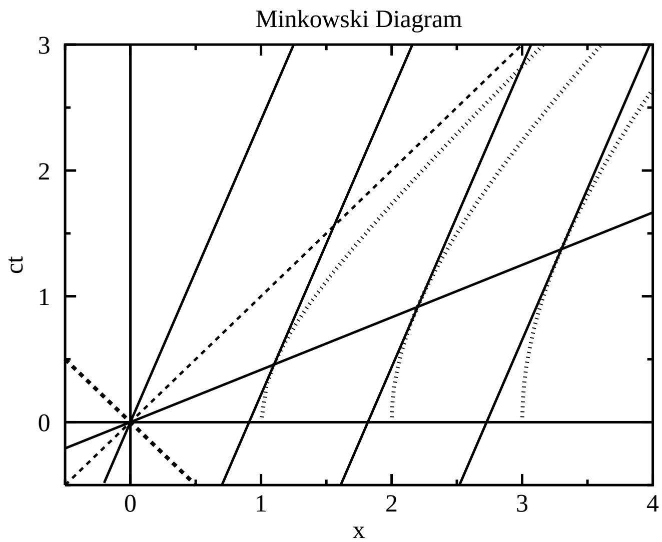
$$\frac{x - x_0}{c} = \int \frac{dx}{c} = \int \frac{\alpha t\, dt}{\sqrt{1 + (\alpha t)^2}} = \frac{1}{\alpha}\left(\sqrt{1 + (\alpha t)^2} - 1\right)$$

If we choose $x_0 = c/\alpha$, the result is particularly simple:

$$\begin{aligned} x &= \frac{c}{\alpha}\sqrt{1 + (\alpha t)^2} \\ x^2 &= \left(\frac{c}{\alpha}\right)^2 + (ct)^2 \\ x^2 - (ct)^2 &= x_0^2 = (c/\alpha)^2 \end{aligned}$$

(While this choice of x_0 makes for nice equations, it typically results in an origin rather far from the object. For example, if $a_0 = g$, x_0 is nearly a light year.)

Since the lhs is an invariant form, we immediately know the form of this equation in boosted frames S'. Additionally note that $\dot{x} = c^2 t/x$ for any α. Thus regardless of α, such hyperbolic objects crossing the line $t/x =$ constant share a common speed. If we boost to that frame, we find the line $t/x =$ constant is the line $t' = 0$. Because of the invariant form, objects on this line must also have $x' = x_0$ and $u' = 0$. Thus if we look at a collection of hyperbolic objects (each with differing x_0 and hence differing α), in *any* standard boosted frame S', at $t' = 0$ we will find the objects at rest with exactly the same x' as they had in the initial frame. This provides the best possible example of a 'rigid', accelerating body.



The dotted lines denote hyperbolic motion of the form:

$$x^2 - (ct)^2 = x_0^2$$

where $x_0 = 1, 2, 3$ is the position at $t = 0$. Also shown is the line $t' = 0$ for some frame S' and the lines $x' = 0, 1, 2, 3$. Note the tangency of the hyperbolas and the $x' = 1, 2, 3$ lines. At $t' = 0$ the hyperbolic objects are at rest in S'.

There are nice expressions for this motion in terms of proper time (i.e., time measured with a clock that moves with the object, and hence subject to time dilation).

$$\begin{aligned}
d\tau &= \frac{dt}{\gamma} = \frac{dt}{\sqrt{1 + (\alpha t)^2}} \\
\tau &= \int \frac{dt}{\sqrt{1 + (\alpha t)^2}} = \frac{1}{\alpha}\ \sinh^{-1}(\alpha t) \\
\sinh(\alpha\tau) &= \alpha t \\
x &= x_0 \sqrt{1 + (\alpha t)^2} = x_0 \cosh(\alpha\tau) \\
ct &= x_0 \sinh(\alpha\tau) \\
\beta &= \frac{\alpha t}{\sqrt{1 + (\alpha t)^2}} = \tanh(\alpha\tau) \\
\gamma &= \sqrt{1 + (\alpha t)^2} = \cosh(\alpha\tau) \\
\mathbb{U} &= \gamma(u, ic) = c(\sinh(\alpha\tau), i\cosh(\alpha\tau)) \\
\mathbb{A} &= \frac{d\mathbb{U}}{d\tau} = c\alpha(\cosh(\alpha\tau), i\sinh(\alpha\tau)) \\
\mathbb{A}^2 &= (c\alpha)^2(\cosh^2(\alpha\tau) - \sinh^2(\alpha\tau)) = a_0^2
\end{aligned}$$