Derivation of the Geodesic Equation and Defining the Christoffel Symbols

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We begin with the line element

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{1}$$

where $g_{\alpha\beta}$ is the metric with $\alpha, \beta = 0, 1, 2, 3$. Also, we are using the Einstein summation convention in which we sum over repeated indices which occur as a subscript and superscript pair. In order to find the geodesic equation, we use the variational principle which states that freely falling test particles follow a path between two fixed points in spacetime which extremizes the proper time, τ.

The proper time is defined by $d\tau^2 = -ds^2$. (We are assuming that $c = 1$.) So, formally, we have

$$\tau_{AB} = \int_A^B \sqrt{-ds^2} = \int_A^B \sqrt{-g_{\alpha\beta} dx^\alpha dx^\beta}.$$

In order to write this as an integral that we can compute, we consider a parametrized worldline, $x^\alpha = x^\alpha(\sigma)$, where the parameter $\sigma = 0$ at point A and $\sigma = 1$ at point B. Then, we write

$$\tau_{AB} = \int_0^1 \left[-g_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \right]^{1/2} d\sigma \equiv \int_0^1 L \left[\frac{dx^\alpha}{d\sigma}, x^\alpha \right] d\sigma. \tag{2}$$

Here we have introduced the Lagrangian, $L\left[\frac{dx^\alpha}{d\sigma}, x^\alpha\right]$.

We note also that

$$L = \frac{d\tau}{d\sigma}.$$

Therefore, for functions $f = f(\tau(\sigma))$, we have

$$\frac{df}{d\sigma} = \frac{df}{d\tau}\frac{d\tau}{d\sigma} = L\frac{df}{d\tau}.$$

We will use this later to change derivatives with respect to our arbitrary parameter σ to derivatives with respect to the proper time, τ.

Using variational methods as seen in classical dynamics, we obtain the Euler-Lagrange equations in the form

$$-\frac{d}{d\sigma}\left(\frac{\partial L}{\partial(dx^\gamma/d\sigma)}\right)+\frac{\partial L}{\partial x^\gamma}=0. \tag{3}$$

We carefully compute these derivatives for the general metric. First we find

$$\begin{aligned}\frac{\partial L}{\partial x^\gamma} &= -\frac{1}{2L}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}\\ &= -\frac{L}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}.\end{aligned} \tag{4}$$

The σ derivatives have been converted to τ derivatives.

Now we compute

$$\begin{aligned}\frac{\partial L}{\partial(dx^\gamma/d\sigma)} &= -\frac{1}{2L}g_{\alpha\beta}\left(\frac{dx^\beta}{d\sigma}\delta_{\alpha\gamma}+\frac{dx^\alpha}{d\sigma}\delta_{\beta\gamma}\right)\\ &= -\frac{1}{2L}\left(g_{\gamma\beta}\frac{dx^\beta}{d\sigma}+g_{\alpha\gamma}\frac{dx^\alpha}{d\sigma}\right)\\ &= -\frac{1}{L}g_{\alpha\gamma}\frac{dx^\alpha}{d\sigma}.\end{aligned} \tag{5}$$

In the last step we used the symmetry of the metric and the fact that α and β are dummy indices.

We differentiate the last result to obtain

$$\begin{aligned}-\frac{d}{d\sigma}\left(\frac{\partial L}{\partial(dx^\gamma/d\sigma)}\right) &= \frac{d}{d\sigma}\left(\frac{1}{L}g_{\alpha\gamma}\frac{dx^\alpha}{d\sigma}\right)\\ &= L\frac{d}{d\tau}\left(g_{\alpha\gamma}\frac{dx^\alpha}{d\tau}\right)\\ &= L\left[g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2}+\frac{dg_{\alpha\gamma}}{d\tau}\frac{dx^\alpha}{d\tau}\right]\\ &= L\left[g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2}+\frac{\partial g_{\alpha\gamma}}{\partial x^\beta}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}\right]\\ &= L\left[g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2}+\frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta}+\frac{\partial g_{\gamma\beta}}{\partial x^\alpha}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right].\end{aligned} \tag{6}$$

Again, we have used the symmetry of the metric and the reindexing of repeated indices. Also, we have eliminated appearances of L by changing to derivatives with respect to the proper time.

So far we have found that

$$\begin{aligned}0 &= -\frac{d}{d\sigma}\left(\frac{\partial L}{\partial(dx^\gamma/d\sigma)}\right)+\frac{\partial L}{\partial x^\gamma}\\ &= L\left[g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2}+\frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta}+\frac{\partial g_{\gamma\beta}}{\partial x^\alpha}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right]-\frac{L}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}.\end{aligned} \tag{7}$$

Rearranging the terms on the right hand side and changing the dummy index α to δ, we have

$$
\begin{aligned}
g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2} &= \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} - \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\gamma\beta}}{\partial x^\alpha}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \\
&= -\frac{1}{2}\left[\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\gamma\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right]\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \\
&= -\frac{1}{2}\left[\frac{\partial g_{\delta\gamma}}{\partial x^\beta} + \frac{\partial g_{\gamma\beta}}{\partial x^\delta} - \frac{\partial g_{\delta\beta}}{\partial x^\gamma}\right]\frac{dx^\delta}{d\tau}\frac{dx^\beta}{d\tau} \\
&\equiv -g_{\alpha\gamma}\Gamma^\alpha_{\delta\beta}\frac{dx^\delta}{d\tau}\frac{dx^\beta}{d\tau}. \qquad (8)
\end{aligned}
$$

We have found the geodesic equation,

$$
\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\delta\beta}\frac{dx^\delta}{d\tau}\frac{dx^\beta}{d\tau} \qquad (9)
$$

where the Christoffel symbols satisfy

$$
g_{\alpha\gamma}\Gamma^\alpha_{\delta\beta} = \frac{1}{2}\left[\frac{\partial g_{\gamma\delta}}{\partial x^\beta} + \frac{\partial g_{\gamma\beta}}{\partial x^\delta} - \frac{\partial g_{\delta\beta}}{\partial x^\gamma}\right]. \qquad (10)
$$

This is a linear system of equations for the Christoffel symbols. If the metric is diagonal in the coordinate system, then the computation is relatively simple as there is only one term on the left side of Equation (10). In general, one needs to use the matric inverse of $g_{\alpha\beta}$. Also, you should note that the Christoffel symbol is symmetric in the lower indices,

$$
\Gamma^\alpha_{\delta\beta} = \Gamma^\alpha_{\beta\delta}
$$

We can solve for the Christoffel symbols by introducing the inverse of the metric, $g^{\mu\gamma}$, satisfying

$$
g^{\mu\gamma}g_{\alpha\gamma} = \delta^\mu_\alpha. \qquad (11)
$$

Here, δ^μ_α is the Kronecker delta, which vanishes for $\mu \neq \alpha$ and is one otherwise. Then,

$$
g^{\mu\gamma}g_{\alpha\gamma}\Gamma^\alpha_{\delta\beta} = \delta^\mu_\alpha\Gamma^\alpha_{\delta\beta} = \Gamma^\mu_{\delta\beta}. \qquad (12)
$$

Therefore,

$$
\Gamma^\mu_{\delta\beta} = \frac{1}{2}g^{\mu\gamma}\left[\frac{\partial g_{\gamma\delta}}{\partial x^\beta} + \frac{\partial g_{\gamma\beta}}{\partial x^\delta} - \frac{\partial g_{\delta\beta}}{\partial x^\gamma}\right]. \qquad (13)
$$