$x = x(u,v)$
$y = y(u,v)$
... inverse...
$u = u(x,y)$
$v = v(x,y)$

$\vec{e}_u = \partial_u \langle x,y \rangle$
$\vec{e}_v = \partial_v \langle x,y \rangle$
generic
$\vec{e}_i = \partial_i \langle x,y \rangle$

$\vec{e}^u = \vec{\nabla} u = \langle \partial_x u, \partial_y u \rangle$
$\vec{e}^v = \vec{\nabla} v = (\partial_x v, \partial_y v)$
generic
$\vec{e}^i = \vec{\nabla} u^i$

$$\left.\begin{aligned} \vec{e}_u \cdot \vec{e}^u &= \frac{\partial x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial u}{\partial y} = \frac{\partial u}{\partial u} = 1 \\ \vec{e}_u \cdot \vec{e}^v &= \frac{\partial x}{\partial u}\frac{\partial v}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial v}{\partial y} = \frac{\partial v}{\partial u} = 0 \end{aligned}\right\} \text{generic } \vec{e}^i \cdot \vec{e}_j = \delta_{ij} \to \delta^i_j$$

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \frac{\partial}{\partial u}\begin{pmatrix} x \\ y \end{pmatrix} du + \frac{\partial}{\partial v}\begin{pmatrix} x \\ y \end{pmatrix} dv = \vec{e}_u\, du + \vec{e}_v\, dv$$

$$ds^2 = \begin{pmatrix} dx \\ dy \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix} = \left[\vec{e}_u\, du + \vec{e}_v\, dv\right] \cdot \left[\vec{e}_u\, du + \vec{e}_v\, dv\right]$$

$$[du, dv] \begin{bmatrix} \vec{e}_u \cdot \vec{e}_u & \vec{e}_u \cdot \vec{e}_v \\ \vec{e}_u \cdot \vec{e}_v & \vec{e}_v \cdot \vec{e}_v \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix}$$

generic: $g_{ij} = \vec{e}_i \cdot \vec{e}_j$

$$\left.\begin{aligned} \vec{A} &= A^u \vec{e}_u + A^v \vec{e}_v \\ &= A_u \vec{e}^u + A_v \vec{e}^v \end{aligned}\right\}$$

note in general $\vec{e}_i$ will depend on location so the same $\vec{A}$ will have different (A^u, A^v) & (A_u, A_v)

Note $A_u = \vec{e}_u \cdot \vec{A} = \vec{e}_u \cdot (\vec{e}_u A^u + \vec{e}_v A^v) = \vec{e}_u \cdot \vec{e}_u A^u + \vec{e}_u \cdot \vec{e}_v A^v$

$$= (g_{ij}) \begin{pmatrix} A^u \\ A^v \end{pmatrix} = \underbrace{g_{ui} A^i}_{\text{sum}}$$

Note $A^u = \vec{e}^u \cdot \vec{A} = \vec{e}^u \cdot (A_u \vec{e}^u + A_v \vec{e}^v) = \vec{e}^u \cdot \vec{e}^u A_u + \vec{e}^u \cdot \vec{e}^v A_v$

$$= \begin{pmatrix} e^u \cdot e^u & e^u \cdot e^v \\ e^u \cdot e^v & e^v \cdot e^v \end{pmatrix} \begin{pmatrix} A_u \\ A_v \end{pmatrix} \to \underbrace{g^{ui} A_i}_{\text{sum}}$$

$$g^{ij} = [g_{ij}]^{-1}$$

Summary: $ds^2 = g_{ij}\, du^i du^j$ (sums) (7.1)

$$A_i = g_{ij} A^j \qquad (7.8)$$

$$A^i = g^{ij} A_j \qquad (7.20)$$

Transformation properties - what defines a tensor

consider alternative coordinate sets u^i & $u^{i'}$

Define $a^{i'}_j = \frac{\partial u^{i'}}{\partial u^j}$ & $a^i_{j'} = \frac{\partial u^i}{\partial u^{j'}}$

Note that these are inverse matrices of each other as

$$\sum_j \frac{\partial u^{a'}}{\partial u^j} \frac{\partial u^j}{\partial u^{b'}} = a^{a'}_j a^j_{b'} = \frac{\partial u^{a'}}{\partial u^{b'}} = \delta^{a'}_{b'}$$

in terms of coordinate differentials:

$$du^{i'} = \frac{\partial u^{i'}}{\partial u^j} du^j = a^{i'}_j du^j$$

Hence if we have an invariant $d\tau$

$$\frac{du^{i'}}{d\tau} = a^{i'}_j \frac{du^j}{d\tau}$$

For the gradient of an invariant

$$\frac{\partial \phi}{\partial u^{i'}} = \frac{\partial u^j}{\partial u^{i'}} \frac{\partial \phi}{\partial u^j} = a^j_{i'} \phi_{,j}$$

The gradient of a vector requires more care as changing axes direction also changes component values - see Covariant Derivative that follows

In terms of our vectors $\vec{e}_i = \langle \partial_i x, \partial_i y \rangle$ & $\vec{e}^i = \vec{\nabla} u^i$ ...

$$\vec{e}_i = \langle \partial_i x, \partial_i y \rangle = \langle \frac{\partial u^{j'}}{\partial u^i} \partial_{j'} x, \frac{\partial u^{j'}}{\partial u^i} \partial_{j'} y \rangle = a^{j'}_i \langle \partial_{j'} x, \partial_{j'} y \rangle = a^{j'}_i \vec{e}_{j'}$$

For $\vec{e}^i$ lets consider a particular vector $\vec{e}^u$ in a 2d (u,v) case were $u(u',v')$ & $u'(x,y)$ & $v'(x,y)$

$$\vec{e}^u = \vec{\nabla} u(u'(x,y), v'(x,y)) = \langle \frac{\partial u}{\partial u'} \frac{\partial u'}{\partial x} + \frac{\partial u}{\partial v'} \frac{\partial v'}{\partial x}, \frac{\partial u}{\partial u'} \frac{\partial u'}{\partial y} + \frac{\partial u}{\partial v'} \frac{\partial v'}{\partial y} \rangle$$

$$= \frac{\partial u}{\partial u'} \vec{e}^{u'} + \frac{\partial u}{\partial v'} \vec{e}^{v'} = a^u_{j'} \vec{e}^{j'}$$

of course same follows for $\vec{e}^v$, so in general

$$\vec{e}^i = a^i_{j'} \vec{e}^{j'}$$

Parallel Transport $\quad \vec{A} = A^i \vec{e}_i$ (note sum) $\longrightarrow \delta u^j \underbrace{\partial_j \vec{e}_i}_{}$

$$0 = \delta\vec{A} = \delta A^i \vec{e}_i + A^i \delta\vec{e}_i$$

this is a vector - expand it in terms of $\vec{e}_k$

so $\quad \delta A^k \vec{e}_k = -A^i \delta u^j \Gamma^k_{ij} \vec{e}_k$

$$\delta A^k = -A^i \delta u^j \Gamma^k_{ij} \quad (7.27)$$

$$\partial_j \vec{e}_i \equiv \Gamma^k_{ij} \vec{e}_k$$

affine connection
Christoffel symbol
aka $\left\{ {k \atop ij} \right\}$

Note: $\Gamma^k_{ij} = \Gamma^k_{ji}$ as $\partial_j \vec{e}_i = \partial_j \partial_i \langle x, y \rangle$

Express in terms of $g_{ij} = \vec{e}_i \cdot \vec{e}_j$

$$\partial_k g_{ij} = (\partial_k \vec{e}_i)\cdot\vec{e}_j + \vec{e}_i\cdot(\partial_k \vec{e}_j)$$

$$= \Gamma^m_{ik} \vec{e}_m \cdot \vec{e}_j + \vec{e}_i \cdot \Gamma^m_{jk} \vec{e}_m = \Gamma^m_{ik} g_{mj} + \Gamma^m_{jk} g_{mi}$$

Aim to isolate <u>one</u> Γ by add/subtract permutations

$$\partial_k g_{ij} = \Gamma^m_{ik} g_{mj} + \Gamma^m_{jk} g_{im}$$

$$+ \partial_i g_{jk} = \Gamma^m_{ij} g_{km} + \Gamma^m_{ki} g_{jm}$$

$$-(\partial_j g_{ki} = \Gamma^m_{kj} g_{jm} + \Gamma^m_{jk} g_{km})$$

$$\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ki} = 2\Gamma^m_{ik} g_{mj}$$

$$\tfrac{1}{2} g^{\ell j} (\qquad\qquad) = \Gamma^\ell_{ik} \quad (7.35)$$

Consider 2 vectors A^i & B_i ; parallel transport both, same dot

$$0 = \delta(A^i B_i) = A^i \delta B_i + B_i \delta A^i = -\Gamma^i_{jk} A^j \delta u^k B_i + A^i \delta B_i$$

$$\delta B_j A^j = \Gamma^\ell_{jk} A^j B_\ell \delta u^k$$

$$\delta B_j = \Gamma^\ell_{jk} B_\ell \delta u^k \quad (7.37)$$

The "real" change in a vector: DA^i is the apparent change, dA^i less the change that is just due to axes change: $\delta A^i = -A^k \delta u^j \Gamma^i_{kj}$

$$DA^i = dA^i + \Gamma^i_{kj} A^k \delta u^j$$

Let the vector A^i be the tangent vector $\frac{du^i}{d\tau}$ to a curve

we seek the "real" change in that vector as we advance along the curve: $\delta u^j = \frac{du^j}{d\tau} d\tau$

$$D\frac{du^i}{d\tau} = d\frac{du^i}{d\tau} + \Gamma^i_{kj} \frac{du^k}{d\tau} \frac{du^j}{d\tau} d\tau$$

If the curve is straight that real change is zero

$$\frac{d}{d\tau}\frac{du^i}{d\tau} + \Gamma^i_{kj} \frac{du^k}{d\tau}\frac{du^j}{d\tau} = 0 \qquad (7.36)$$

The online file "geodesic.pdf" shows an alternative derivation using Lagrangian & Calculus of Variations -
There this same result comes from minimizing the path length.

Covariant Derivative – seek $D_i A_j = a_i^{i'} a_j^{j'} D_{i'} A_{j'}$

$$\partial_i A_j = a_i^{i'} \partial_{i'} (a_j^{j'} A_{j'}) = \underbrace{a_i^{i'} a_j^{j'} \partial_{i'} A_{j'}}_{\text{good}} + \underbrace{a_i^{i'} A_{j'} \partial_{i'} (a_j^{j'})}_{\text{bad}}$$

$$\partial_j \vec{e_i} = \Gamma_{ij}^{k} \vec{e_k}$$

$$= a_j^{j'} \partial_{j'} (a_i^{i'} \vec{e}_{i'}) = a_j^{j'} \vec{e}_{i'} \partial_{j'} a_i^{i'} + a_j^{j'} a_i^{i'} \partial_{j'} \vec{e}_{i'}$$

$\vec{e}_{i'} = a_{i'}^{k} \vec{e_k}$ $\qquad$ $\partial_{j'} \vec{e}_{i'} = \Gamma_{i'j'}^{k'} \vec{e}_{k'}$, $\quad \vec{e}_{k'} = a_{k'}^{k} \vec{e}_k$

$$\partial_i A_j = a_i^{i'} a_j^{j'} \partial_{i'} A_{j'} + a_i^{i'} A_{j'} (\partial_{i'} a_j^{j'})$$

$$\Gamma_{ij}^{k} A_k = a_j^{j'} A_{i'} \partial_{j'} a_i^{i'} + a_j^{j'} a_i^{i'} \Gamma_{i'j'}^{k'} A_{k'}$$

rename i' $\qquad$ rename j'

$$\partial_i A_j - \Gamma_{ij}^{k} A_k = a_i^{i'} a_j^{j'} (\partial_{i'} A_{j'} - \Gamma_{i'j'}^{k'} A_{k'}) \quad ✓ \qquad (7.44)$$

$$A_{j;i}$$ semicolon

seek $D_i A^j = a^{i'}_i a^j_{j'} D_{i'} A^{j'}$

$$\partial_i A^j = a^{i'}_i \partial_{i'}(a^j_{j'} A^{j'}) = a^{i'}_i a^j_{j'} \partial_{i'} A^{j'} + a^{i'}_i A^{j'} \partial_{i'}(a^j_{j'})$$

Given: $$\Gamma^k_{ij} = a^{j'}_j \underbrace{a^k_{i'} \partial_{j'} a^{i'}_i}_{= -a^{i'}_i \partial_{j'} a^k_{i'}} + a^{j'}_j a^{i'}_i \Gamma^{k'}_{i'j'} a^k_{k'}$$

contract A^i (but call it k) $j \to i$ $k \to j$

$$A^k \Gamma^j_{ki} = a^{j'}_i \left(- a^{i'}_k A^k \partial_{j'} a^j_{i'}\right) + a^{j'}_i a^{i'}_k A^k \Gamma^{k'}_{i'j'} a^j_{k'}$$

$A^{i'}$

call it j'

call it i'

$$A^k \Gamma^j_{ki} = a^{i'}_i \left(- A^{j'} \partial_{i'} a^j_{j'}\right) + a^{i'}_i A^{j'} \Gamma^{k'}_{j'i'} a^j_{k'}$$

$$\partial_i A^j = a^{i'}_i A^{j'} \partial_{i'} a^j_{j'} + a^{i'}_i a^j_{j'} \partial_{i'} A^{j'}$$

$$\partial_i A^j + \Gamma^j_{ik} A^k = a^{i'}_i a^j_{j'} \left[\partial_{i'} A^{j'} + \Gamma^{j'}_{i'k'} A^{k'} \right] \quad (7.43)$$

$\equiv A^j_{;i}$ semi colon