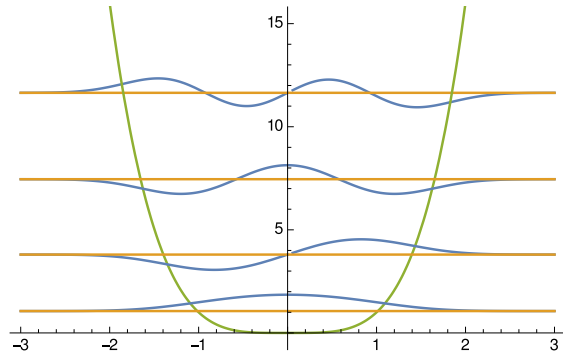Consider Schrödinger's equation for a quartic potential in dimensionless coordinates:

$$-\psi'' + x^4\psi \;=\; E\psi \qquad\qquad (1)$$
$$\psi'' \;=\; -\big[E - x^4\big]\,\psi \qquad (2)$$

See right a stacked wavefunction plot of the eigenfunctions.

The exact eigenenergies are: $(1.0604, 3.7997, 7.4557, 11.6447)$. In the following problems you are to obtain approximations to those eigenenergies using *Mathematica* (i.e., I believe the calculations are too ugly to do by hand). Please turn in enough *Mathematica* code that I can see exactly what commands you used. (There is no reason to turn in *Mathematica* plots.)

1. (RR) Note that the below trial wavefunctions depend on $|x|$ (which is a non-analytic function); it might be best to properly restrict the domain to positive x and then $|x| \to x$. Note that the below trial wavefunctions have two adjustable parameters (b & c). *Mathematica* can still find a minimum with code like
   `FindMinimum[e,{{b,guess1},{c,guess2}}]`
   Good guesses can be found by displaying contour plots with code like:
   `ContourPlot[e,{b,b`$_{min}$`,b`$_{max}$`},{c,c`$_{min}$`,c`$_{max}$`}]`

   (a) Use the trial wavefuntion:
   $$f(x) = \exp(-b|x|^c) \qquad\qquad (3)$$
   Why is this a reasonable trial wavefuntion? Report your eigeneenrgy estimate. Your result should be a bit above 1.0604. Why?

   (b) Use the trial wavefuntion:
   $$f(x) = x\exp(-b|x|^c) \qquad\qquad (4)$$
   Why is this a reasonable trial wavefunction? Report your eigeneenrgy estimate. Your result should be a bit above 3.7997. Why?

2. (WKB) Write down the <u>definite</u> integral (include integral limits) that is the lhs of the WKB approximation. The integral should depend on the energy E. Have *Mathematica* plot the integral as a function of E. Write down rhs of the WKB approximation: a simple formula that involves π and the whole number n that describes the number of half-wavelengths in the allowed region. Using this plot, visually estimate the value of E that produces the required multiple of π. Find precise values with *Mathematica* code like:
   `FindRoot[wkb/Pi-`*whatever*`,{e,`*your estimate*`}]`
   Find by this method the first four eigenenergies and compare your results to the exact answers given above. If your results are not greater than the exact values should you be concerned? Remark: If you used `e` in your *Mathematica* code before starting this problem you may need to: `ClearAll[e]`.