#1 (dTIPT) As stated the states ψ_{21} & ψ_{12} are degenerate [they have $(n_x, n_y) = (2,1)$ & $(1,2)$ and the energy depends on $n_x^2 + n_y^2 = 5$ for both states]

We must calculate the 2x2 matrix of V on this degenerate subspace. Note that this is relatively easy as for $V = \lambda\, \delta(x - L/4)\, \delta(y - 2L/3)$

$$\iint f(x,y)\, V(x,y)\, dx\, dy = \lambda\, f\left(\tfrac{L}{4}, \tfrac{2L}{3}\right)$$

For us $f(x,y) = \psi_{21}^2,\ \psi_{21}\psi_{12},\ \psi_{12}^2$

$$\langle \psi_{21} | V | \psi_{21} \rangle = \lambda\, \psi_{21}^2 \Big|_{\substack{x = L/4 \\ y = 2L/3}} = \lambda \left(\frac{2}{L}\right)^2 \left[\sin\left(\frac{2\pi}{L}\frac{L}{4}\right) \sin\left(\frac{1\pi}{L}\frac{2L}{3}\right)\right]^2$$

$\sin\left(\frac{2\pi}{L}\frac{L}{4}\right) \to 1$, $\quad \sin\left(\frac{1\pi}{L}\frac{2L}{3}\right) \to \sqrt{3}/2$

$$= \lambda \left(\frac{2}{L}\right)^2 \frac{3}{4}$$

$$\langle \psi_{21} | V | \psi_{12} \rangle = \lambda\, \psi_{21}\psi_{12} \Big|_{\substack{x = L/4 \\ y = 2L/3}} = \lambda \left(\frac{2}{L}\right)^2 \left[\begin{array}{l} \sin\left(\frac{2\pi}{L}\frac{L}{4}\right) \sin\left(\frac{1\pi}{L}\frac{2L}{3}\right) \times \\ \sin\left(\frac{1\pi}{L}\frac{L}{4}\right) \sin\left(\frac{2\pi}{L}\frac{2L}{3}\right) \end{array}\right]$$

1, $\quad \frac{1}{\sqrt{2}}$, $\quad \frac{\sqrt{3}}{2}$, $\quad -\frac{\sqrt{3}}{2}$

$$= -\lambda \left(\frac{2}{L}\right)^2 \frac{3}{4\sqrt{2}}$$

$$\langle \psi_{12} | V | \psi_{12} \rangle = \lambda\, \psi_{12}^2 \Big|_{\substack{x = L/4 \\ y = 2L/3}} = \lambda \left(\frac{2}{L}\right)^2 \left[\sin\left(\frac{\pi}{L}\frac{L}{4}\right) \sin\left(\frac{2\pi}{L}\frac{2L}{3}\right)\right]^2$$

$\sin\left(\frac{\pi}{L}\frac{L}{4}\right) \to \frac{1}{\sqrt{2}}$, $\quad \sin\left(\frac{2\pi}{L}\frac{2L}{3}\right) \to -\frac{\sqrt{3}}{2}$

$$= \lambda \left(\frac{2}{L}\right)^2 \frac{3}{8}$$

$$\text{matrix } V = \lambda \frac{3}{L^2} \begin{bmatrix} 1 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}$$

seek eigenvalues: $(1-x)\left(\frac{1}{2}-x\right) - \frac{1}{2} = 0 \to x^2 - \frac{3}{2}x = 0 \to x = \begin{cases} 0 \\ 3/2 \end{cases}$

eigenvectors: $0 \leftrightarrow \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$ $\qquad \frac{3}{2} \leftrightarrow \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$

if desired divide by $\sqrt{3}$ to normalize

$\psi_{21} + \sqrt{2}\,\psi_{12}$ $\qquad -\sqrt{2}\,\psi_{21} + \psi_{12}$

$E_1 = 0$ $\qquad E_1 = \lambda \frac{3}{L^2} \cdot \frac{3}{2} = \frac{\lambda}{L^2}\frac{9}{2}$

exam #3 (TDPT) $\quad \psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$

$$E_1 = \langle \psi_n | V | \psi_n \rangle = \lambda \int_{L/2-b}^{L/2+b} \frac{2}{L} \sin^2\left(\frac{n\pi x}{L}\right) dx$$

n=1 even
n=2 odd
n=3 even
etc

define even/odd about center of box:

if $\psi_m V \psi_n$ is odd integral is zero

so m & n must both be even or both be odd for matrix element $\langle \psi_m | V | \psi_n \rangle$ to be non zero

In general ψ_n that are odd (ie n = even) are zero at $x = L/2$ which is where V is non zero. As a result generally $\langle \overset{n}{\text{even}} | V | \overset{n}{\text{even}} \rangle < \langle \overset{n}{\text{odd}} | V | \overset{n}{\text{odd}} \rangle$

2ND order term of shift of ground state (n=1)

$$\frac{|\langle n=3 | V | n=1 \rangle|^2}{\frac{\hbar^2 \pi^2}{2mL^2}(1^2 - 3^2)} = \frac{\left| \int_{L/2-b}^{L/2+b} \psi_1(x) \psi_3(x)\, dx \right|^2}{\frac{\hbar^2 \pi^2}{2mL^2}(1^2 - 3^2)}$$

exam #4 (TIPT) Note the wavefunctions $\psi_{n_x n_y} = |n_x n_y\rangle$

are orthonormal ie: $\langle n_x n_y | n_x n_y \rangle = 1$

$\langle n_x' n_y' | n_x n_y \rangle = 0$ if either $n_x' \neq n_x$, $n_y' \neq n_y$

this makes it easy to do integrals as since λ = constant it can be pulled out of integral

so: $\langle n_x' n_y' | V | n_x n_y \rangle = \lambda \langle n_x' n_y' | n_x n_y \rangle$

$$= \lambda \begin{cases} 0 & \text{if } n_x' \neq n_x \text{ or } n_y' \neq n_y \\ 1 & \text{if } n_x' = n_x \text{ \& } n_y' = n_y \end{cases}$$

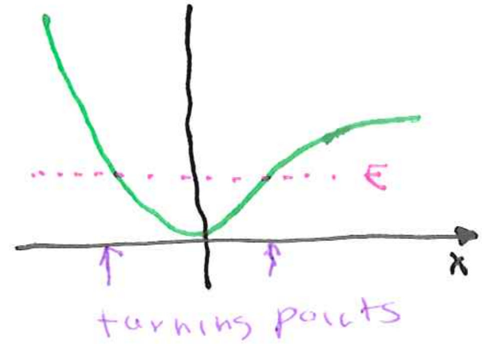
Thus all second & third order matrix elements are zero. Only first order is nonzero

$$E_1 = \langle n_x n_y | V | n_x n_y \rangle = \lambda$$

WKB morse $\quad k(x) = \alpha\sqrt{E-(1-e^{-x})^2}$

turning points in terms of x not super easy but they will be easy if we make the suggested substitution: $\quad u = 1-e^{-x}$

$$du = e^{-x}\,dx = (1-u)\,dx$$



$$\pi(n-1/2) = \int k(x)\,dx = \alpha\int\sqrt{E-(1-e^{-x})^2}\,dx$$

$$= \alpha\int\sqrt{E-u^2}\,\frac{du}{1-u}$$

← turning points (where $k=0$) now clearly $\sqrt{E} \equiv A$

$$= \alpha\int_{-A}^{A}\frac{\sqrt{A^2-u^2}}{1-u}\,du = \pi\alpha\left(1-\sqrt{1-A^2}\right) = \pi\alpha\left(1-\sqrt{1-E}\right)$$

seek $E(n)$ ... $\quad (n-1/2) = \left(1-\sqrt{1-E}\right)\alpha$

$$\sqrt{1-E} = 1 - \frac{(n-1/2)}{\alpha}$$

$$1-E = \left(1-\frac{(n-1/2)}{\alpha}\right)^2 = 1 - \frac{2(n-1/2)}{\alpha} + \frac{(n-1/2)^2}{\alpha^2}$$

$$\frac{2(n-1/2)}{\alpha} - \frac{(n-1/2)^2}{\alpha^2} = E$$

Note: our n starts at 1 whereas handout exact form starts at $n=0$ ... these are exactly the same expression