$$\int_{-\infty}^{+\infty} e^{-\alpha x^2-\beta x}\,dx = \sqrt{\frac{\pi}{\alpha}}\,e^{\beta^2/4\alpha} \qquad \int_{-\infty}^{+\infty} x^2 e^{-\alpha x^2}\,dx = \frac{-d}{d\alpha}\int_{-\infty}^{+\infty} e^{-\alpha x^2}\,dx \qquad \int_0^\infty x^n\,e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}} \qquad \int f(x)\delta(x-a)\,dx = f(a)$$

$$H\psi = i\hbar\partial_t\psi \qquad H\psi = E\psi \qquad H = \frac{p^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\,\partial_x^2 + V(x) \qquad p = -i\hbar\partial_x \qquad [p,x] = -i\hbar$$

Particle-in-a-box with $V(x)=0$ for $0<x<L$, but $V(x)=\infty$ elsewhere

$$E_n = \frac{(\hbar k)^2}{2m} \qquad \psi_n(x) = \sqrt{\frac{2}{L}}\,\sin(kx) \quad \text{where} \quad k = \frac{n\pi}{L} \qquad n = 1,2,3\ldots$$

3-d: $|n_x n_y n_z\rangle = \psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z)$ $\quad E = \dfrac{(\hbar k)^2}{2m}$ where $\vec{\mathbf{k}} = (n_x\pi/L_x, n_y\pi/L_y, n_z\pi/L_z)$

spherical box: $R(r) = j_\ell(kr)$ $\qquad E = \dfrac{(\hbar k)^2}{2m}$ $\qquad$ where: $kR =$ zero of j_ℓ

Harmonic Oscillator with $V(x) = \dfrac{1}{2}\,m\omega^2 x^2$ $\qquad E_n = \hbar\omega\,(n+\tfrac{1}{2})$ $\qquad n = 0,1,2\ldots$

$|n\rangle = \psi_n(x) = N_n H_n(\xi)\,e^{-\frac{1}{2}\xi^2}$ $\quad$ where $\quad \xi = \sqrt{\dfrac{m\omega}{\hbar}}\,x$ $\quad$ and H_n is an n^{th} degree polynomial

$$a_- = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p}{m\omega}\right) = \frac{1}{\sqrt{2}}\,(\xi + \partial_\xi) \qquad a_+ = a_-^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - i\frac{p}{m\omega}\right) = \frac{1}{\sqrt{2}}\,(\xi - \partial_\xi) \qquad x = \sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-)$$

$$[a_-, a_+] = 1 \qquad [H, a_\pm] = \pm\hbar\omega a_\pm \qquad H = \hbar\omega\left(\tfrac{1}{2} + a_+a_-\right) \qquad a_-\,|n\rangle = \sqrt{n}\,|n-1\rangle \qquad a_+\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

2-d: $(n_x + n_y + 1) = (2n_r + |m| + 1)$ $\qquad$ 3-d: $\left(n_x + n_y + n_z + \tfrac{3}{2}\right) = \left(2n_r + \ell + \tfrac{3}{2}\right)$

Angular Momentum: $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$ $\qquad [L_i, V_j] = i\hbar\epsilon_{ijk}V_k$ where vector $\vec{\mathbf{V}} = \vec{\mathbf{r}}, \vec{\mathbf{p}}, \vec{\mathbf{L}}$ $\qquad |\ell m\rangle = Y_{\ell m}(\theta,\phi)$ $\quad -\ell \le m \le +\ell$

$$\vec{\mathbf{L}}^2\,|\ell m\rangle = \ell(\ell+1)\,\hbar^2\,|\ell m\rangle \qquad L_z\,|\ell m\rangle = m\hbar\,|\ell m\rangle \qquad L_\pm\,|\ell m\rangle = \sqrt{\ell(\ell+1) - m(m\pm 1)}\;\hbar\,|\ell\, m\pm 1\rangle$$

$$L_\pm = L_x \pm iL_y \qquad [L_+, L_-] = 2\hbar L_z \qquad [L_z, L_\pm] = \pm\hbar L_\pm \qquad [\vec{\mathbf{L}}^2, L_\pm] = 0 \qquad [L_i, \vec{\mathbf{V}}\cdot\vec{\mathbf{W}}] = 0$$

Spin $\frac{1}{2}$: $\quad \vec{\mathbf{S}} = \dfrac{\hbar}{2}\,\vec{\sigma}$ $\qquad |\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \chi_+ = \uparrow = \begin{pmatrix}1\\0\end{pmatrix}$ $\qquad |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle = \chi_- = \downarrow = \begin{pmatrix}0\\1\end{pmatrix}$

Clebsch-Gordan: $|jm\rangle = \sum C(jm;\ell m_\ell, sm_s)\,|\ell m_\ell\rangle\,|sm_s\rangle$ $\qquad$ know how to use table!

Radial Equation: $\psi(r,\theta,\phi) = Y_{\ell m}(\theta,\phi)R(r)$ $\qquad R(r) = \dfrac{u(r)}{r}$

$$\left[\frac{-\hbar^2}{2m}\left(\partial_r^2 + \frac{2}{r}\partial_r\right) + \frac{\hbar^2\ell(\ell+1)}{2mr^2} + V(r)\right] R = E\,R \qquad \left[\frac{-\hbar^2}{2m}\partial_r^2 + \frac{\hbar^2\ell(\ell+1)}{2mr^2} + V(r)\right] u = E\,u$$

H atom: $\quad H = \dfrac{p^2}{2m} - \dfrac{Ze^2}{4\pi\epsilon_0 r}$ $\qquad E_n = -\dfrac{1}{2}\,mc^2\,\dfrac{(Z\alpha)^2}{n^2} = -\dfrac{1}{2}\,\dfrac{Z^2e^2}{4\pi\epsilon_0 a_0 n^2} \approx -13.6\text{ eV}\;\dfrac{Z^2}{n^2}$ $\qquad n = 1,2,3,\ldots$

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2} \approx .53\text{ Å} \qquad \alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137} \qquad n = n_r + \ell + 1 \;\therefore\; 0 \le \ell \le n-1 \qquad \rho = \sqrt{\frac{8m|E|}{\hbar^2}}\,r = \frac{2Zr}{na_0}$$

$|n\ell m\rangle = R_{n\ell}(\rho)\,Y_{\ell m}(\theta,\phi)$ $\quad$ where $\quad R_{n\ell} = N_{n\ell}\,\rho^\ell\,L_{n_r}^{2\ell+1}(\rho)\,e^{-\frac{1}{2}\rho}$ $\qquad N_{n\ell} = \dfrac{2}{n^2}\sqrt{\dfrac{(n-\ell-1)!}{(n+\ell)!}}$

Approximation Methods:

WKB: $\quad \displaystyle\int k(x)\,dx = (n-\tfrac{1}{2})\pi$ $\quad$ (two linear turning points) $\qquad \hbar k(x) = p(x) = \sqrt{2m(E-V(x))}$

Rayleigh-Ritz: $\quad$ minimize $\quad E = \dfrac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$

Perturbation Theory: $\quad E_n^1 = \langle n|H'|n\rangle$ $\qquad E_n^2 = \displaystyle\sum_{k\neq n}\frac{|\langle k|H'|n\rangle|^2}{E_n^0 - E_k^0}$ $\qquad$ degenerate: diagonalize matrix $\langle i|H'|j\rangle$

Time Dependent: $\quad c_b(t) \simeq -\dfrac{i}{\hbar}\displaystyle\int_0^t H'_{ba}(t')\,e^{i(E_b-E_a)t'/\hbar}\,dt'$

if $H' = V(\mathbf{r})\cos\omega t$ $\quad$ then: $P_{a\to b} \approx \dfrac{|V_{ab}|^2}{\hbar^2}\,\dfrac{\sin^2((\omega_0-\omega)t/2)}{(\omega_0-\omega)^2}$ $\qquad$ E1 Unpolarized Light: $R_{a\to b} = \dfrac{\pi\rho(\omega_0)}{3\epsilon_0\hbar^2}\,|q\langle\psi_b|\vec{\mathbf{r}}|\psi_a\rangle|^2$

Selection Rules: $\quad \Delta m = \pm1, 0 \quad \Delta l = \pm 1$

E1: $\Delta J = 0, \pm1$ $(0 \not\to 0)$, $\quad \Delta M = 0, \pm1$ $(0 \not\to 0$, if $\Delta J = 0)$, $\quad \Delta S = 0$, $\quad \Delta L = 0, \pm1$ $(0 \not\to 0)$

$$\vec{r} = \langle x,y,z\rangle = r\langle\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\rangle = r\sqrt{\frac{2\pi}{3}}\langle -Y_{11} + Y_{1-1}, i(Y_{11} + Y_{1-1}), \sqrt{2}\,Y_{10}\rangle$$

3 Y integral: $\langle Y_{l_3m_3}|Y_{l_2m_2}|Y_{l_1m_1}\rangle = \sqrt{\dfrac{(2l_1+1)(2l_2+1)}{4\pi(2l_3+1)}}\;\langle l_3 0|l_1 l_2 0 0\rangle\langle l_3 m_3|l_1 l_2 m_1 m_2\rangle \propto$ Clebsch-Gordan: $C(l_3m_3; l_2m_2; l_1m_1)$