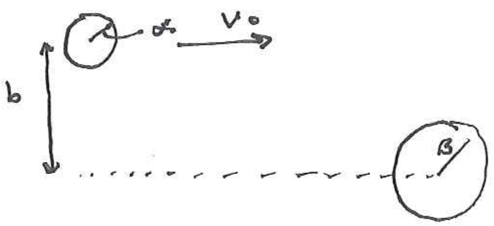
Classical scattering

Lab frame



radii : α & β

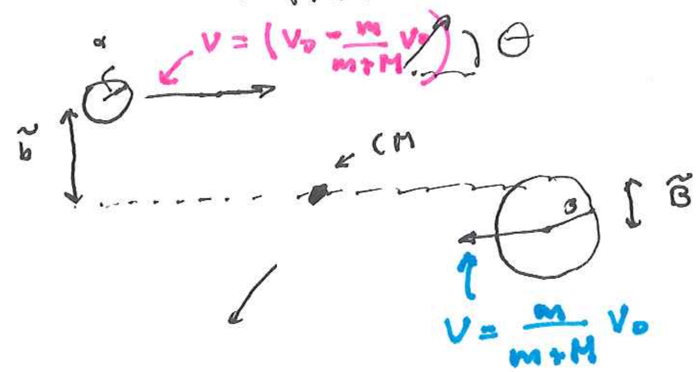
lab speed projectile : v_0

impact parameter b

angular momentum = $m b v_0$

$$V_{CM} = \frac{m}{m+M} v_0$$

CM fram



$\tilde{b} + \tilde{B} = b$

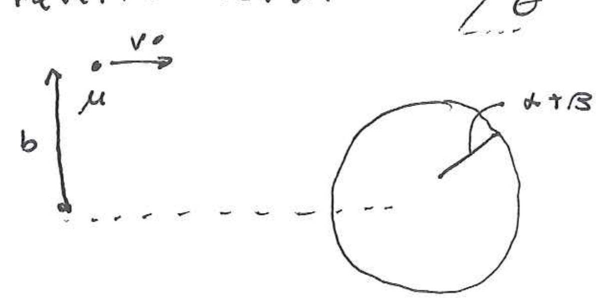
angular momentum

$$= m \tilde{b} \left(\frac{M}{m+M}\right) v_0 + M \tilde{B} \left(\frac{m}{m+M}\right) v_0$$

$$= \mu b v_0$$

Note: the scattering angle in lab frame ≠ scattering angle in CM frame. Its a longish problem to derive relationship between the two - which we totally avoid (ie skip)

relative coordinate

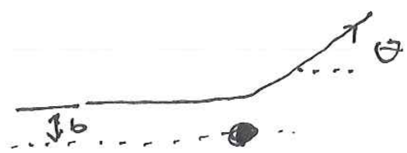


(total) cross-section: from the point of view of the projectile the force-range of the target (or the combined force range of projectile & target) extends out some distance. If it misses those blocked-out areas there will be no scattering. The blocked area [$\pi(\alpha+\beta)^2$ in above example] is the cross-section.

Note: if the force range is infinite [as in $\frac{1}{r^2}$] then there will always be some scatter - but the scatter angle will be small if the force is small. There are additional mathematical complications in the long-range force case which we avoid (ie, skip) while scatter/no scatter provides some information (total cross-section) the angular distribution of the projectile provides additional information (differential cross-section)

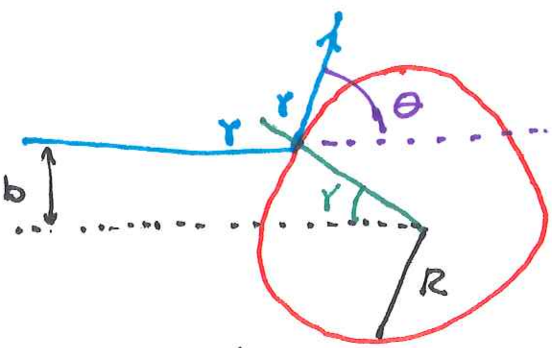
Again from the point of view of the projectile, the aim point determines the force experienced and hence the final deflection. Color the cross-section of the target according to what final deflection will result (like point values on a bulls-eye:)

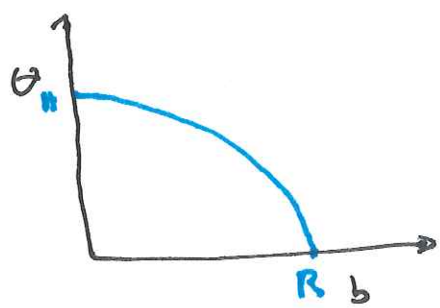
back scatter : $170-180°$
back-scatter $160-170°$
back-scatter $150-160°$
back-scatter $100-150°$
back scatte $60-100°$
back scatt $0-60°$



while we've expressed above in terms of bins — in fact there will be a functional relationship : $\theta(b)$

Example: hard-sphere scattering : $R = \alpha + \beta$





$$\sin\gamma = \frac{b}{R}$$

$$\theta = \pi - 2\gamma$$

$$\gamma = \frac{\pi - \theta}{2}$$

$$\sin\gamma = \sin\left(\frac{\pi-\theta}{2}\right) = \cos\left(\frac{\theta}{2}\right)$$

$\sin\gamma = b/R$

$$2\cos^{-1}\left(\frac{b}{R}\right) = \theta$$

differential cross-section: Let $d\sigma$ be the bit of area that produces a scattering into the solid angle $d\Omega$

$$D(\theta) = \frac{d\sigma}{d\Omega} \equiv \text{differential scattering cross-section.}$$

Example: area that leads to $d\theta = 2\pi b\, db$

solid angle $d\Omega = 2\pi \sin\theta\, d\theta$

$$\frac{d\sigma}{d\Omega} = \frac{2\pi b\, db}{2\pi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

absolute value as it doesn't matter if $b\uparrow$ produces smaller or larger θ

In terms of the bulls-eye figure $\frac{d\sigma}{d\Omega}$ is the area of a band divided by the $d\Omega$ of the corresponding range of angles.

Eg: $b = R\cos(\theta/2)$

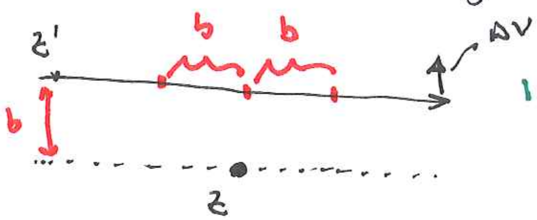
$$\frac{db}{d\theta} = -\frac{R}{2}\sin\left(\frac{\theta}{2}\right)$$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{\underbrace{R\cos\left(\frac{\theta}{2}\right)\frac{R}{2}\sin\left(\frac{\theta}{2}\right)}_{\frac{1}{2}\sin\left(2\cdot\frac{\theta}{2}\right)}}{\sin\theta}$$

$$= \frac{R^2}{4}$$ (Note: does not depend on θ!)

total cross-section $= \int \frac{d\sigma}{d\Omega}\, d\Omega = \frac{R^2}{4}\int d\Omega = R^2\pi$

Rutherford scattering — crude approximation: cranked projectile.



$$\text{Impulse} \approx F\Delta t = \frac{z' Z e^2}{4\pi\epsilon_0 b^2}\,\frac{2b}{v_0} = m\,\Delta v$$

$$\theta \approx \frac{\Delta v}{v_0} = \frac{z' Z e^2\; 2b}{4\pi\epsilon_0 b^2 \underbrace{m v_0^2}_{2E}} = \frac{z' Z e^2}{4\pi\epsilon_0 b E}$$

$$b = \frac{z' Z e^2}{4\pi\epsilon_0 E\,\theta}$$

$$\left|\frac{db}{d\theta}\right| = \frac{z' Z e^2}{4\pi\epsilon_0 E\,\theta^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\,\frac{db}{d\theta} = \left(\frac{z' Z e^2}{4\pi\epsilon_0 E}\right)^2 \frac{1}{\theta^4}$$

Correct answer: $\theta^4 \to (2\sin\theta/2)^4$

Note: infinite total cross section

$$\sigma \propto \int_0 \frac{1}{\theta^4}\, 2\pi \sin\theta\, d\theta \approx 2\pi \int_0 \frac{1}{\theta^3}\, d\theta$$

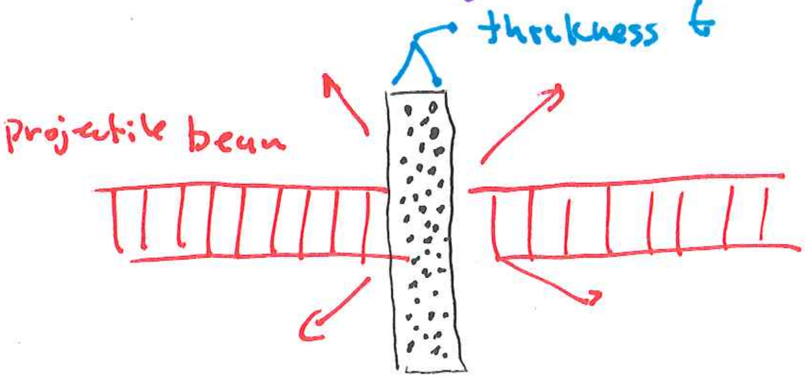
Note: if consider $F \propto \frac{1}{r^3} \Rightarrow \theta \propto \frac{1}{b^2} \Rightarrow \frac{d\sigma}{d\Omega} \sim \frac{1}{\theta^3}$

Remark: the above finds frequent small-angle scattering but recall main point of Rutherford expt was occasional (rare) back-scatter ($\theta = 180°$) which required a nucleus much heavier than α.

Practical scattering.

fraction scattered $= \sigma n_2$

fraction scattered into $d\Omega$
$= \frac{d\sigma}{d\Omega} n_2\, d\Omega$



$$\frac{\#\text{ nuclei}}{\text{volume}} = \underset{\substack{\uparrow\\ \text{density}}}{\rho}\; \frac{1}{\text{mass atom}} \equiv n$$

$$\frac{\#\text{ nuclei}}{\text{area}} = \rho\, t\, \frac{1}{\text{mass atom}} \equiv n_2$$