Relativity: $\underset{T+mc^2}{E^2} - p^2c^2 = (mc^2)^2 \Rightarrow T = \sqrt{(mc^2)^2 + p^2c^2} - mc^2$

$$= mc^2\left(\sqrt{1+\left(\tfrac{p}{mc}\right)^2} - 1\right)$$

$$x = -\left(\tfrac{p}{mc}\right)^2$$

$$\sqrt{1-x} = (1-x)^{-\alpha} = \sum \frac{(\alpha)_n}{n!}x^n = 1 - \frac{1}{2}x - \frac{1}{8}x^2 \qquad \alpha = -1/2$$

$$T = mc^2\left(1 + \frac{1}{2}\left(\frac{p}{mc}\right)^2 - \frac{1}{8}\left(\frac{p}{mc}\right)^4 + \dots\right)$$

$$= mc^2 + \frac{p^2}{2m} \underbrace{- \frac{p^4}{8m^3c^2}} + \dots$$

$$E_1^1 = \frac{-1}{8m^3c^2}\langle p^4\rangle = \frac{-1}{8m^3c^2}\langle \overset{2m(E-V)}{p^2}\psi \,|\, p^2\psi\rangle$$

$$= \frac{-1}{2mc^2}\langle (E-V)^2\rangle = \frac{-1}{2mc^2}\left\{E^2 - 2E\langle V\rangle + \langle V^2\rangle\right\}$$

$$V = \frac{-Ze^2}{4\pi\varepsilon_0 r} \qquad \text{seek } \left\langle \frac{1}{r}\right\rangle = \frac{1}{n^2a} \quad \& \quad \left\langle \frac{1}{r^2}\right\rangle = \frac{1}{(\ell+1/2)n^3a^2}$$

$$E = -\frac{1}{2}\frac{mc^2\alpha^2}{n^2}Z^2$$

$$\frac{e^2}{4\pi\varepsilon_0} = \frac{e^2}{4\pi\varepsilon_0\hbar c}\frac{\hbar c}{mc}mc = \alpha\,\underbrace{\lambda_c}\,mc^2$$

compton wavelength of electron

$$\{\ \} = E^2\left\{1 - \frac{2}{E}\langle V\rangle + \frac{1}{E^2}\langle V^2\rangle\right\}$$

$$= E^2\left\{1 - \underbrace{\frac{4n^2}{mc^2\alpha^2Z^2}Z\alpha mc^2\lambda\frac{1}{n^2a}}_{=4} + \underbrace{\frac{4n^4}{(mc^2\alpha^2Z^2)^2}\frac{Z^2\alpha^2\lambda_c^2m^2c^2}{(\ell+1/2)n^3a^2}}_{\frac{4n}{\ell+1/2}}\right\}$$

$$E_1 = -\frac{E^2}{2mc^2}\left\{\frac{4n}{\ell+1/2} - 3\right\}$$

does not depend on m

Spin-orbit: as viewed by an orbiting electron, there is a magnetic field & therefore $PE = -\vec{\mu}\cdot\vec{B}$

$\vec{\mu} = \frac{q}{2m}\vec{S}$ classically, but "g factor = 2.002" corrects for relativity

From E&M moving thru an electric field - see B:

$$\vec{B}'_\perp = \underbrace{\gamma}_{\text{neglect}}\left[\frac{-\vec{v}\times\vec{E}}{c^2}\right] \qquad \text{(eq 22-40 Reitz, Milford, Christy)}$$

$\vec{E}$ from nucleus: $\frac{Ze}{r^3}\vec{r} \Rightarrow \vec{B}_\perp = \frac{\vec{r}\times\vec{v}}{c^2}\frac{Zek}{r^3} = \frac{\vec{L}}{mc^2}\frac{Zek}{r^3}$

$$PE = -\left[\frac{-e}{2m}g\,\vec{S}\right]\cdot\frac{\vec{L}}{mc^2}\frac{Zek}{r^3} = \frac{Ze^2}{4\pi\varepsilon_0}\frac{g}{2m^2c^2}\frac{1}{r^3}\vec{S}\cdot\vec{L}$$

$k = \frac{1}{4\pi\varepsilon_0}$

"Thomas Precession" $\to g-1 \approx 1$

Note: $\frac{e^2}{4\pi\varepsilon_0\hbar c} \equiv$ fine structure constant $\alpha \approx \frac{1}{137}$

unit check: $Z\frac{e^2}{4\pi\varepsilon_0\hbar c}\,\hbar\,\underbrace{\frac{g-1}{2m^2c}\frac{1}{r^3}\vec{S}\cdot\vec{L}\,\frac{mc^2}{mc^2}}_{\text{units of angular momentum}^3}$

units of angular moment^3 ; $mc^2 \leftarrow$ rest mass Energy

estimate size of $\left(\frac{\hbar}{mcr}\right) \to r \approx n^2 a$; a = Bohr radius $= \frac{1}{\alpha}\lambda_c$

$\frac{\hbar}{mc}$ = "compton wavelength λ_c"

$$\sim Z\alpha(g-1)\left(\frac{\hbar}{mcr}\right)^3 mc^2 \sim \frac{\alpha^4}{n^6}mc^2 Z(g-1)$$

$$H_{SO} = \alpha\left(\frac{\lambda_c}{r}\right)^3\frac{S\cdot L}{\hbar^2}\,mc^2\,\frac{Z(g-1)}{2}$$

Note: $\frac{\vec{L}\cdot\vec{S}}{\hbar^2} = \frac{1}{2\hbar^2}(J^2 - L^2 - S^2) = \frac{1}{2}\left[j(j+1) - l(l+1) - \frac{3}{4}\right]$

J^2: $\hbar^2 j(j+1)$ where $j = l \pm \frac{1}{2}$; L^2: $\hbar^2 l(l+1)$; S^2: $\hbar^2\frac{3}{4}$

$j = l - 1/2$: $(l-\frac{1}{2})(l+1/2) - l^2 - l - \frac{3}{4} = -l-1$

$j = l + 1/2$: $(l+\frac{1}{2})(l+3/2) - l^2 - l - \frac{3}{4} = l$

$$\left\langle\frac{1}{r^3}\right\rangle = \frac{1}{l(l+\frac{1}{2})(l+1)n^3a^3}$$

$$H_{so} = \alpha \left(\frac{\lambda_c}{r}\right)^3 \frac{S\cdot L}{\hbar^2}\, mc^2 \,\frac{\overbrace{Z\,(g-1)}^{1}}{2}$$

$$= \alpha \,\frac{\alpha^3}{l(l+\frac{1}{2})(l+1)\,n^3}\,\frac{1}{4}\begin{Bmatrix} l \\ -l-1 \end{Bmatrix} mc^2$$

cancel $l+1$ in denom

$$\Rightarrow \frac{-1}{l(l+1/2)} = \frac{-1}{(j+1/2)(l+1/2)}$$

cancel l in denom

$$\frac{+1}{(l+\frac{1}{2})(l+1)} = \frac{+1}{(l+\frac{1}{2})(j+1/2)}$$

$$= \frac{\alpha^4}{n^3\,(j+1/2)(l+1/2)}\,\frac{1}{4}\,\{\pm 1\}\, mc^2$$

$$= \frac{E^2 n}{mc^2\,(j+1/2)(l+1/2)}\,\{\pm 1\}$$

Combine with relativity:

$$\frac{E^2 n}{mc^2}\left[\frac{\{\pm 1\}}{(j+\frac{1}{2})(l+1/2)} - \frac{2\cancel{n}}{l+1/2} + \frac{3}{2n}\right]$$

$$\frac{2}{(l+1/2)(j+1/2)}\left[\underbrace{\{\pm\tfrac{1}{2}\} - (\overbrace{j+\tfrac{1}{2}}^{l\pm 1/2})}_{=-(l+1/2)}\right] = \frac{-2}{(j+1/2)}$$

$$= \frac{E^2 n}{mc^2}\left[\frac{-2}{(j+1/2)} + \frac{3}{2n}\right] = -\underbrace{\frac{E^2\cancel{n}\,2}{mc^2}}_{-|E|\frac{\alpha^2}{n^2}}\left[\frac{n}{(j+\frac{1}{2})} - \frac{3}{4}\right]$$