Typically for any problem that has a known solution, WKB will exactly match that solution. Nevertheless WKB should not be trusted for small n.

Fine structure constant $\approx \frac{1}{137}$

Eg - H atom boundstate: $E<0$, $E = -\frac{1}{2}\frac{mc^2\alpha^2}{n^2}$

$n = n_r + \ell + 1$

Convert 3d problem to 1d via $\psi = \frac{u(r)}{r} Y_{\ell m}(\theta,\phi)$ with boundary condition $u(r=0)=0$

$$-\frac{\hbar^2}{2m}u'' + \underbrace{\frac{\hbar^2 \ell(\ell+1)}{2mr^2}}_{\text{"centrifugal" potential}} u - \underbrace{\frac{Ze^2}{4\pi\varepsilon_0 r}}_{\text{attractive electrostatic potential}} u = -|E|u$$

$$-u'' = \frac{2m}{\hbar^2}\left\{\frac{Ze^2}{4\pi\varepsilon_0 r} - \frac{\hbar^2 \ell(\ell+1)}{2mr^2} - |E|\right\} u$$

$$= \frac{2m|E|}{\hbar^2 \cancel{2m} r^2}\underbrace{\left\{\frac{Ze^2}{4\pi\varepsilon_0|E|}r - \frac{\hbar^2(\ell+1/2)^2}{2m|E|} - r^2\right\}}u$$

proper treatment of boundary condition $u(r=0)=0$ results in $\ell(\ell+1)\rightarrow(\ell+1/2)^2$
The difference is just 1/4

This is quadratic $-(r-a)(r-b)$ where linear term $a+b = \frac{Ze^2}{4\pi\varepsilon_0|E|}$ & constant term $ab = \frac{\hbar^2(\ell+1/2)^2}{2m|E|}$

Table: $\int_a^b \frac{1}{x}\sqrt{-(x-a)(x-b)} = \frac{\pi}{2}\left(\sqrt{b}-\sqrt{a}\right)^2 = \frac{\pi}{2}\left(a+b-2\sqrt{ab}\right)$

Above: $-u'' = k^2 u$ where $k = \sqrt{\frac{2m|E|}{\hbar^2}}\frac{1}{r}\sqrt{-(r-a)(r-b)}$

WKB: $\pi(n_r + 1/2) = \int_a^b k(r)\,dr = \sqrt{\frac{2m|E|}{\hbar^2}}\int_a^b \frac{1}{r}\sqrt{-(r-a)(r-b)}\,dr$

$$= \frac{\sqrt{2m|E|}}{\hbar}\frac{\pi}{2}\left(\frac{Ze^2}{4\pi\varepsilon_0|E|} - 2\sqrt{\frac{\hbar^2(\ell+1/2)^2}{2m|E|}}\right)$$

$n_r + \ell + 1 \equiv n$

$$= \frac{\pi}{2}\left(\frac{\sqrt{2mc^2}\,Ze^2}{4\pi\varepsilon_0\hbar c\sqrt{|E|}} - 2(\ell+1/2)\right) \qquad \frac{Ze^2}{4\pi\varepsilon_0\hbar c} = \alpha$$

$$\underbrace{(n_r + 1/2 + \ell + 1/2)}_{n_r+\ell+1 \equiv n} = \sqrt{\frac{mc^2}{2}}\frac{\alpha}{\sqrt{|E|}}$$

$$\sqrt{|E|} = \frac{\sqrt{\frac{mc^2}{2}}\,\alpha}{n}$$

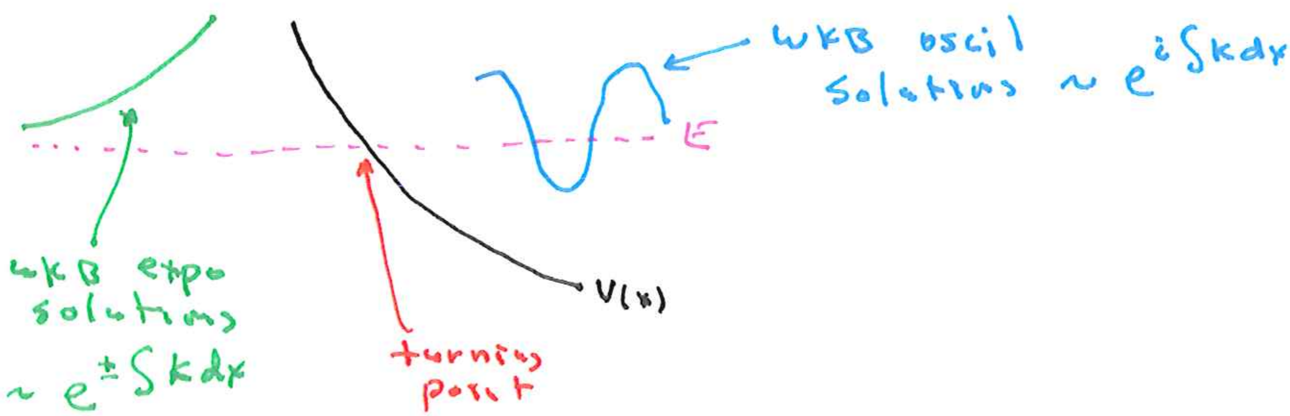
$$|E| = \frac{mc^2}{2}\,\frac{\alpha^2}{n^2} \quad \checkmark$$

In general $k = \sqrt{\frac{2m(E-V)}{\hbar^2}}$ depends on E, so $\int k\,dx$ depends on E. WKB condition reads

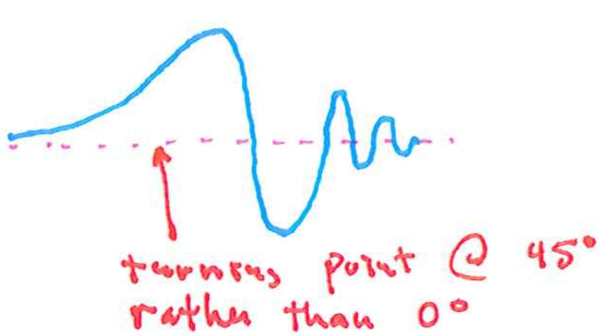
$$\text{function}(E) = \int k\,dx = \pi(n+1/2)$$

Invert this result to get $E = \text{function}(n)$

WKB is valid away from turning points where $E=V$



How connect solutions across turning point? Approx V(x) as a linear function which has known solution Ai(x) ... connect oscil part of Ai with $e^{i\int k dx}$ connect expo part of Ai with $e^{\int k dx}$ ... much calculations ... result at turning point its as if WKB solution was at 45° — half way to zero



Remark: If have "hard" $V=\infty$ turning point then $\psi=0$ there ... ie WKB starts at $\theta=0°$

with a "hard" ($V=\infty$) boundary we need integer # of half wavelengths (just like standing waves in organ pipe) $\Longrightarrow$ $\int \underbrace{k\,dx}_{2\pi \frac{dx}{\lambda} = 2\pi\text{(fraction of wavelength)}} = \pi n$

$n = 1, 2, 3$ # $\frac{\lambda}{2}$ between turning points

seek the sum of wavelength fractions to be $\frac{\text{integer}}{2}$

If you have "linear" turning points — you start at $45^\circ = \frac{1}{8}\lambda$ so

$$\underbrace{\frac{1}{2\pi}\int k\,dx}_{\text{total wavelengths between turning points}} + \frac{1}{8} \times \underbrace{[\#\text{ linear turning pts}]}_{0,\,1,\,2} = \frac{n}{2}$$

$n = 1, 2, 3$

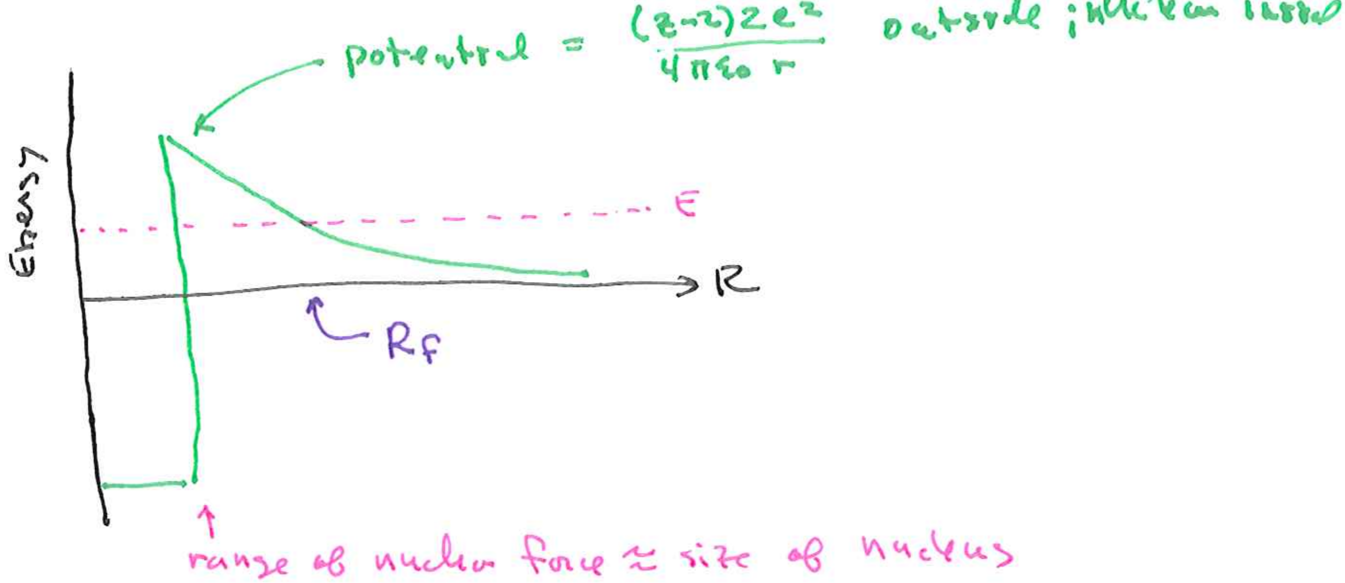
0 — both hard; 2 — both linear

$$\frac{1}{\pi}\int k\,dx + \frac{1}{4} \times [\#\text{ linear}] = n$$

$$\int k\,dx = \pi\left(n - \frac{1}{4}[\#\text{ linear}]\right)$$

typically this is 2 so $n - 1/2$. You'll often find this in books as $n + 1/2$. The difference is whether n starts at zero or one

Another common application of WKB is tunneling eg in α decay (Gamow)



Classically an α inside the nucleus (or outside) cannot transit the disallowed region. Nevertheless we measure αs leaving nucleus (α decay) with an energy well below the barrier. Also we can cause nuclear reactions with αs that have too little energy to approach the nucleus. In the disallowed region ψ has expo decay — net reduction $e^{-\int_R^{R_f} K\,dx} = e^{-\gamma}$ as prob $\propto |\psi|^2 \rightarrow e^{-2\gamma}$. One can actually calculate the integral and compare to half-lives

The fit while not perfect explains a huge amount of the variation in half life.

Remark: There are lots of cases where can measure tunneling eg in e^- emission from cold cathode much as in 370 lab Thermionic Emission.