Fourier transform - Convolution thm

$$h(x) = \int f(x-y)\, g(y)\, dy$$

$$H(k) = \int e^{-ikx} h(x)\, dx = \int \underbrace{e^{-ikx}}_{e^{-ik(x-y)}\ e^{-iky}} f(x-y)\, g(y)\, dy\, dx$$

$$= \int e^{-ik(x-y)} f(x-y)\, dx \int e^{-iky} g(y)\, dy$$

$$= F(k)\, G(k)$$

$$\int V(r)\, e^{i\vec{q}\cdot\vec{r}}\, d^3r = \underbrace{\frac{1}{4\pi\epsilon_0} \int \frac{1}{r} e^{i\vec{q}\cdot\vec{r}}\, dr}_{\text{Rutherford } \frac{d\sigma}{d\Omega}} \times \underbrace{\int \rho(r)\, e^{i\vec{q}\cdot r}\, dr}_{F(\vec{q})}$$

$$\uparrow \quad \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3r'$$

remark: put Z of ρ into Rutherford so $\int \rho(r)\, d^3r = 1$

Taylor expand $e^{i\vec{q}\cdot\vec{r}} = 1 + i\,\vec{q}\cdot\vec{r} - \frac{1}{2} \sum q_i q_j r_i r_j + \ldots$

$$\int \rho\, \vec{r}\, d^3r = 0 \qquad \int \rho\, r_i r_j = \delta_{ij} \frac{1}{3} \langle r^2 \rangle$$

$$F(q) = 1 - \frac{1}{6} q^2 \langle r^2 \rangle$$ ← "charge radius"

$$\rho = \begin{cases} \frac{3}{4\pi R^3} \\ 0 \end{cases} \rightarrow F = 3\left(\frac{\sin(kR)}{(kR)^3} - \frac{\cos(kR)}{(kR)^2} \right)$$

Fourier Trans

① unit sphere $\rho = \frac{3}{4\pi R^3}$ ← $q = 2k\sin\frac{\theta}{2}$ eventually

$$FT = \frac{3}{4\pi R^3}\int_0^R 1\, e^{i\vec{k}\cdot\vec{r}}\, r^2 dr\, dc\, d\phi$$

$d\phi \to 2\pi$, $dc \to \frac{e^{ikr}-e^{-ikr}}{ikr}$

$$= \frac{3}{kR^3}\int_0^R \sin(kr)\, r\, dr = \frac{3}{k^3R^3}\int_0^{kR}\sin(u)\, u\, du$$

$$= \frac{3}{k^3R^3}\left[\sin u - u\cos u\right]_0^{kR} = 3\left[\frac{\sin(kR)}{k^3R^3} - \frac{\cos(kR)}{k^2R^2}\right]$$

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Note: $\frac{\sin x}{x} - \cos x = \left(1 - \frac{x^2}{3!}\right) - \left(1 - \frac{x^2}{2!}\right) = \frac{x^2}{3} \to F(0) = 1$

next order: $\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!}\right) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right) = -\frac{x^4}{30}$

$\to F = 1 - \frac{1}{10}(kR)^2$

Note $\langle r^2\rangle = \frac{\int r^4 dr\, 4\pi}{\int r^2 dr\, 4\pi} = \frac{\frac{1}{5}}{\frac{1}{3}} R^2 = \frac{3}{5}R^2$

$\frac{1}{6}\langle r^2\rangle = \frac{1}{10}R^2$

② Yukawa / Rutherford $V = \frac{ZZ'e^2}{4\pi\varepsilon_0}\frac{e^{-\mu r}}{r}$

$$FT = \frac{ZZ'e^2}{4\pi\varepsilon_0}\int_0^\infty e^{-\mu r}\, r\, e^{i\vec{k}\cdot\vec{r}}\, dr\, dc\, d\phi$$

$d\phi \to 2\pi$, $dc \to \frac{e^{ikr}-e^{-ikr}}{ikr}$

$$= \frac{ZZ'e^2}{\varepsilon_0 k}\int e^{-\mu r}\sin(kr)\, dr$$

$\mathrm{Im}\left(e^{-(\mu - ik)r}\right) \to \mathrm{Im}\left[\frac{1}{\mu - ik}\right] = \frac{k}{\mu^2+k^2}$

$$= \frac{ZZ'e^2}{\varepsilon_0 k^2}$$

$x\, \frac{m}{2\pi\hbar^2} \longrightarrow -\frac{ZZ'e^2}{4\pi\varepsilon_0\, \frac{k^2\hbar^2}{2m}} \longrightarrow \frac{-ZZ'e^2}{4\pi\varepsilon_0 E\,(2\sin\theta/2)^2}$

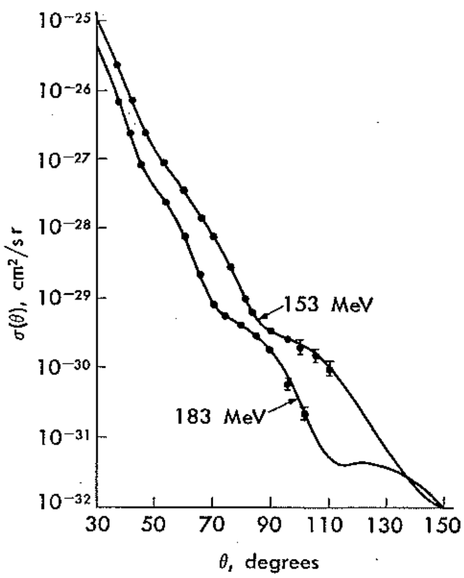

Fig. 4-5 The best-fitting Fermi density for gold gives this excellent agreement with experiment. (After Hofstadter [5].)

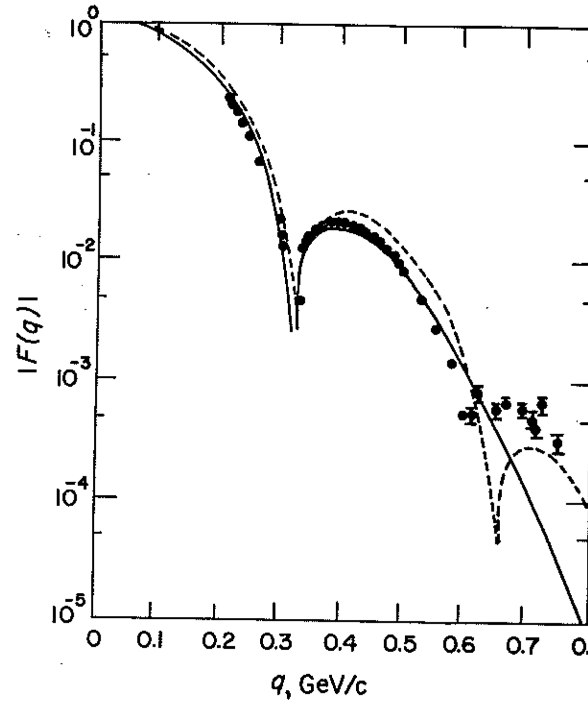

Fig. 4-8 Electron elastic-scattering form factor for ^{16}O. The experimental data of McCarthy and Sick [24] are shown by dots. The full line is the theoretical fit obtained by assuming a closed-shell configuration and using harmonic-oscillator wave functions; the dashed curve is obtained from Woods-Saxon wave functions. (After Donnelly and Walker [25].)

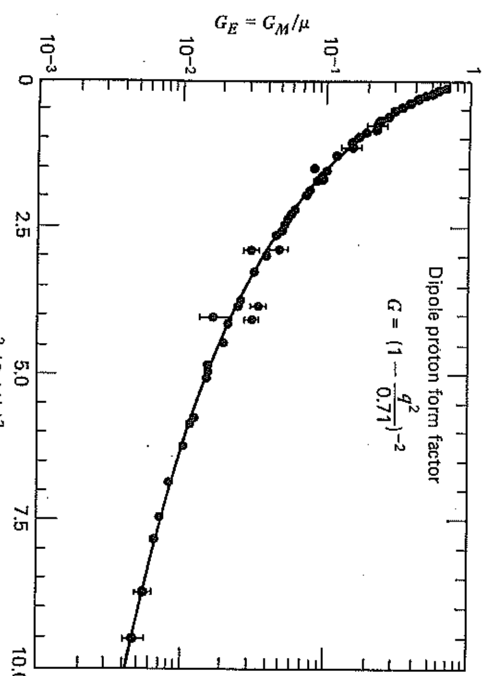

Fig. 8.4 The proton form factors as a function of q^2.

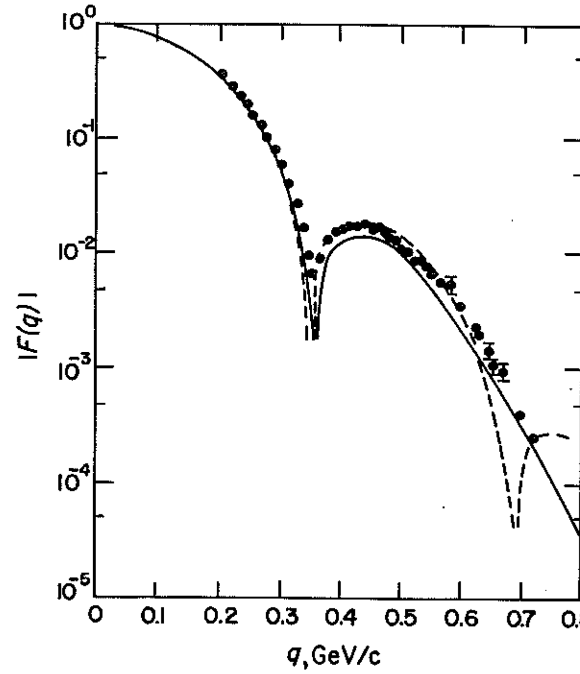

Fig. 4-9 Same as in Fig. 4–8, but for ^{12}C. (After Donnelly and Walker [25].)

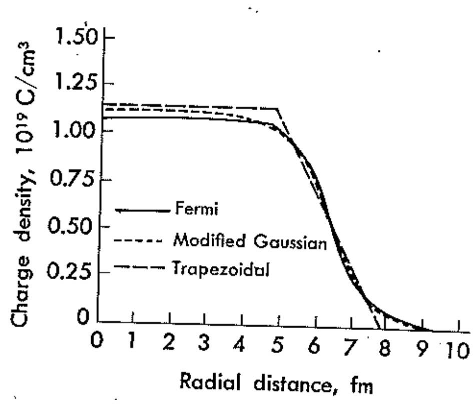

Fig. 4-6 Densities for gold, all of which fit the electron-scattering data. (After Hofstad [5].)