General considerations:

match point: $R \propto \cos\delta_\ell \, j_\ell(kr) - \sin\delta_\ell \, y_\ell(kr)$

"log derivative" $\dfrac{R'}{R} \equiv \gamma = \dfrac{k(\cos\delta \, j' - \sin\delta \, y')}{\cos\delta \, j - \sin\delta \, y}$

$$\frac{\gamma}{k}\left(j - \tan\delta \, y\right) = j' - \tan\delta \, y'$$

$$\tan\delta = \frac{j' - \frac{\gamma}{k} j}{y' - \frac{\gamma}{k} y}$$

since as $kr \to 0$ js $\to 0$ but ys $\to \infty \Rightarrow \tan\delta = 0$

$\delta = n\pi$

unless zero in denominator — "zero energy resonance"

since $\sigma = \frac{4\pi}{k^2} \sum (2\ell+1) \sin^2\delta_\ell$ $\delta_\ell \neq n\pi$ as $k \to 0$

means ∞ cross-section as $k \to 0$!

$\ell = 0$ is usual suspect — $y_0(z) \approx \frac{-1}{z}$

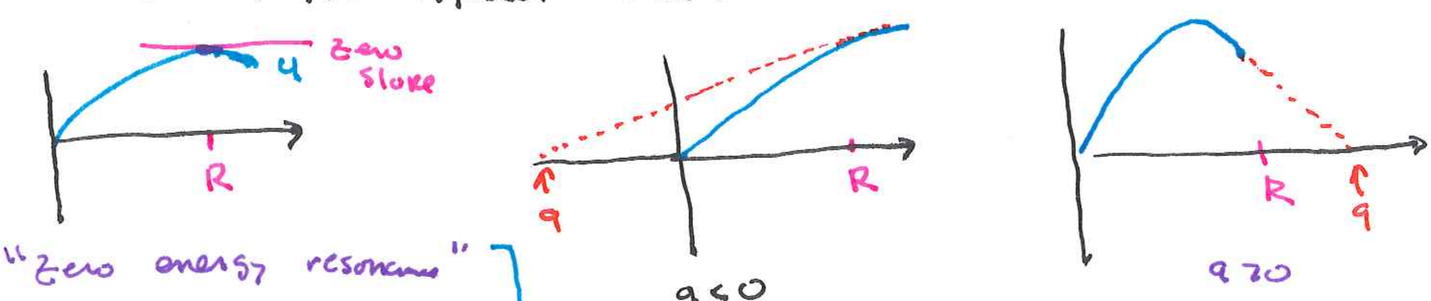
compare u & R forms of wavefunction: $R = \frac{u}{r}$, $R' = \frac{-u}{r^2} + \frac{u'}{r}$ $\Big\}$ $\frac{R'}{R} = \frac{-1}{r} + \frac{u'}{u}$

if $u' = 0$, $\gamma = \frac{-1}{r}$

denom $= y' + \frac{1}{kr} y = \frac{1}{(kr)^2} - \frac{1}{(kr)^2} = 0$!

otherwise taylor expand $\tan\delta \approx -ak$ ← scattering length.

Note $a > 0 \Rightarrow$ slope < 0



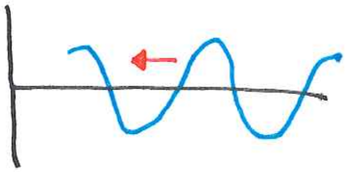
"zero energy resonance"

"$\not\infty$ scattering length" ⎬ same can occur with multiple nodes inside R

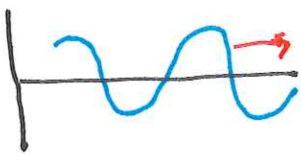
Note: scattering length NOT directly connected to size of potential but IS connected to total cross-section $\sigma \approx 4\pi a^2$. In case of hard sphere $R = a$ [note 4x area!]

Note: this $\ell = 0$ scattering has isotropic $\frac{d\sigma}{d\Omega}$

$$\cos\delta\, j_\ell - \sin\delta\, y_\ell \rightarrow \frac{1}{kr}\cos\left(kr - \frac{\pi}{2}(\ell+1) + \delta\right)$$



attractive potentials "suck" in phase $\delta > 0$; [same value kr further along in cycle]



repulsive potentials "push" out phase $\delta < 0$; [same value of kr not as far thru cycle]

π's of phase sucked in (at k=0) tells you the # nodes for zero energy solution = total # bound states for that value of ℓ.

For finite depth potentials, at sufficiently large KE; PE "small" → little effect, so $\delta \to 0$

Note: for ∞ hard sphere we are always missing the phase (kR) that would be between $r=0$ & $r=R$ → δ goes increasingly negative.

we expanded $\tan\delta_0$ with one term in Taylor series

$$\frac{1}{\tan\delta_0} = \cot\delta_0 = \frac{1}{-ka} + \frac{1}{2} r_0 k + \dots$$

effective range

$$1 + \cot^2\delta = \frac{1}{\sin^2\delta} \quad \& \quad \sigma = \frac{4\pi}{k^2}\sin^2\delta_0$$

$$= \frac{4\pi}{k^2\left[1 + \left(\frac{-1}{ka} + \frac{1}{2} r_0 k\right)^2\right]}$$

$$= \frac{4\pi a^2}{\left[k^2a^2 + \left(-1 + \frac{1}{2} r_0 a k^2\right)^2\right]}$$

r_0 can be approx related to potential's range.

For resonances we've seen δ_ℓ jump past a $\pi/2$ —
Taylor expand! $\cot\delta = \frac{E-E_r}{\Gamma/2}$

$$\Rightarrow \sigma_\ell = \frac{4\pi(2\ell+1)}{k^2}\frac{1}{1+\cot\delta^2} = \frac{4\pi(2\ell+1)}{k^2}\frac{(\Gamma/2)^2}{(E-E_R)^2+(\Gamma/2)^2}$$

[Note Γ = full width at half-max; $\lambda = \frac{\Gamma}{\hbar}$ is decay rate for resonance ie $e^{-\lambda t}$]

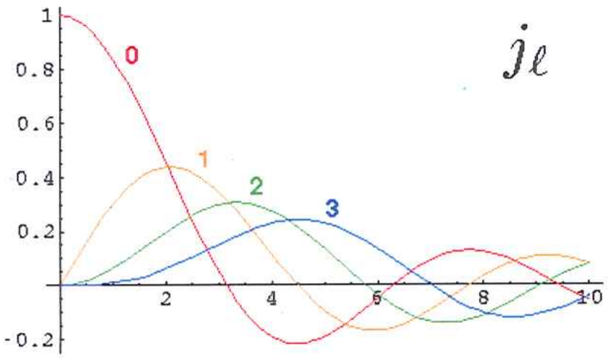
Breit-Wigner aka Lorentz "line width"

optical Thm: $f = \sum \frac{(2\ell+1)}{k} e^{i\delta_\ell} \sin\delta_\ell P_\ell(\cos\theta)$

Consider $\operatorname{Im} f(\theta=0) = \sum \frac{(2\ell+1)}{k}\sin^2\delta_\ell \cdot 1$

$$= \frac{k}{4\pi}\sigma_{total}$$

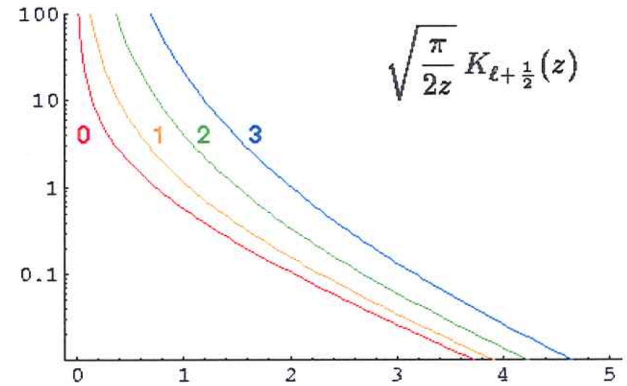
Note: as $k\uparrow$ more ℓs required [$kR \sim \ell$ to turn on]
so increasingly difficult to calculate.
Further $k\uparrow$ (ie $\lambda\downarrow$) should be required to resolve things at small distance scales.
So: partial waves mostly useful at "low" energy



$$j_0(z) = \frac{\sin z}{z}$$

$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}$$

$$j_2(z) = \left(\frac{3}{z^3} - \frac{1}{z}\right)\sin z - \frac{3}{z^2}\cos z$$



$$j_l(z)\begin{cases} \xrightarrow[z\to 0]{} & \dfrac{(\frac{1}{2})!}{(l+\frac{1}{2})!}\left(\dfrac{z}{2}\right)^l = \dfrac{1}{(\frac{3}{2})_l}\left(\dfrac{z}{2}\right)^l \\ \xrightarrow[z\to\infty]{} & \dfrac{1}{z}\cos\left(z - \frac{\pi}{2}(l+1)\right) \end{cases}$$

$$R(\rho) = N j_l(\rho) = N\,\frac{(\rho/2)^l(\frac{1}{2})!}{(l+\frac{1}{2})!}\,{}_0F_1\left(\begin{matrix} - \\ l+\frac{3}{2}\end{matrix}; -\frac{\rho^2}{4}\right)$$

$$j_l(z) = (\tfrac{1}{2})!\sqrt{\frac{2}{z}}\,J_{l+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}\,J_{l+\frac{1}{2}}(z)$$

$$\left(-\frac{\partial^2}{\partial\rho^2} - \frac{2}{\rho}\frac{\partial}{\partial\rho} + \frac{l(l+1)}{\rho^2}\right)R(\rho) = \pm R(\rho)$$

$$-\left(\frac{1}{\rho^2}\,\partial_\rho\rho^2\partial_\rho R\right) + \frac{l(l+1)}{\rho^2}\,R = \pm R$$

$$-(\rho R)'' + \frac{l(l+1)}{\rho^2}\,\rho R = \pm\rho R$$

$$\sqrt{\frac{\pi}{2z}}\,K_{l+\frac{1}{2}}(z)\begin{cases} \xrightarrow[z\to 0]{} & (\frac{1}{2})!^2\,\dfrac{(l-\frac{1}{2})!}{(-\frac{1}{2})!}\left(\dfrac{2}{z}\right)^{l+1} = \dfrac{\pi}{4}\,(\tfrac{1}{2})_l\left(\dfrac{2}{z}\right)^{l+1} \\ \xrightarrow[z\to\infty]{} & \dfrac{\pi}{2z}\,e^{-z} \end{cases}$$

$$\sqrt{\frac{\pi}{2z}}\,K_{0+\frac{1}{2}}(z) = \frac{\pi}{2z}\,e^{-z}$$

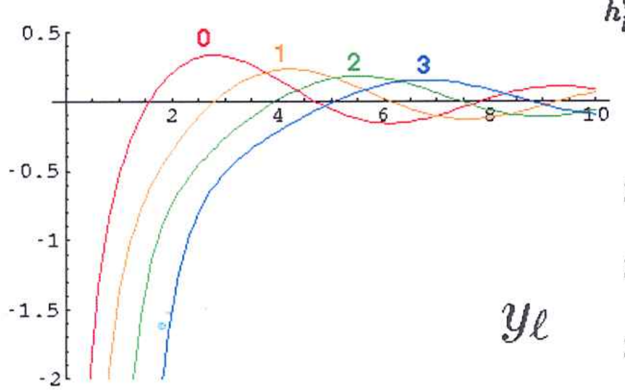
$$\sqrt{\frac{\pi}{2z}}\,K_{1+\frac{1}{2}}(z) = \frac{\pi}{2z}\,e^{-z}\left(1+\frac{1}{z}\right)$$

$$\sqrt{\frac{\pi}{2z}}\,K_{2+\frac{1}{2}}(z) = \frac{\pi}{2z}\,e^{-z}\left(1+\frac{3}{z}+\frac{3}{z^2}\right)$$

$$y_l(z)\begin{cases} \xrightarrow[z\to 0]{} & -\dfrac{(l-\frac{1}{2})!}{2(-\frac{1}{2})!}\left(\dfrac{2}{z}\right)^{l+1} = (-\tfrac{1}{2})_{l+1}\left(\dfrac{2}{z}\right)^{l+1} \\ \xrightarrow[z\to\infty]{} & \dfrac{1}{z}\sin\left(z - \frac{\pi}{2}(l+1)\right) \end{cases}$$

$$\sqrt{\frac{\pi}{2z}}\,I_{l+\frac{1}{2}}(z)\begin{cases} \xrightarrow[z\to 0]{} & \dfrac{(\frac{1}{2})!}{(l+\frac{1}{2})!}\left(\dfrac{z}{2}\right)^l = \dfrac{1}{(\frac{3}{2})_l}\left(\dfrac{z}{2}\right)^l \\ \xrightarrow[z\to\infty]{} & \dfrac{1}{2z}\,e^z \end{cases}$$

$$h_l^{(1,2)}(z) = j_l(z) \pm i y_l(z) \xrightarrow[z\to\infty]{} \frac{1}{z}\,e^{\pm i\left(z - \frac{\pi}{2}(l+1)\right)}$$



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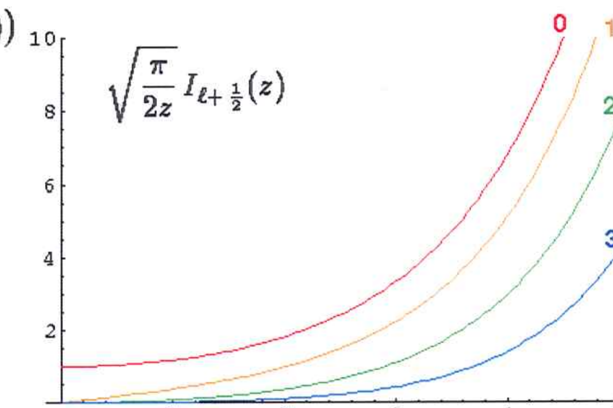
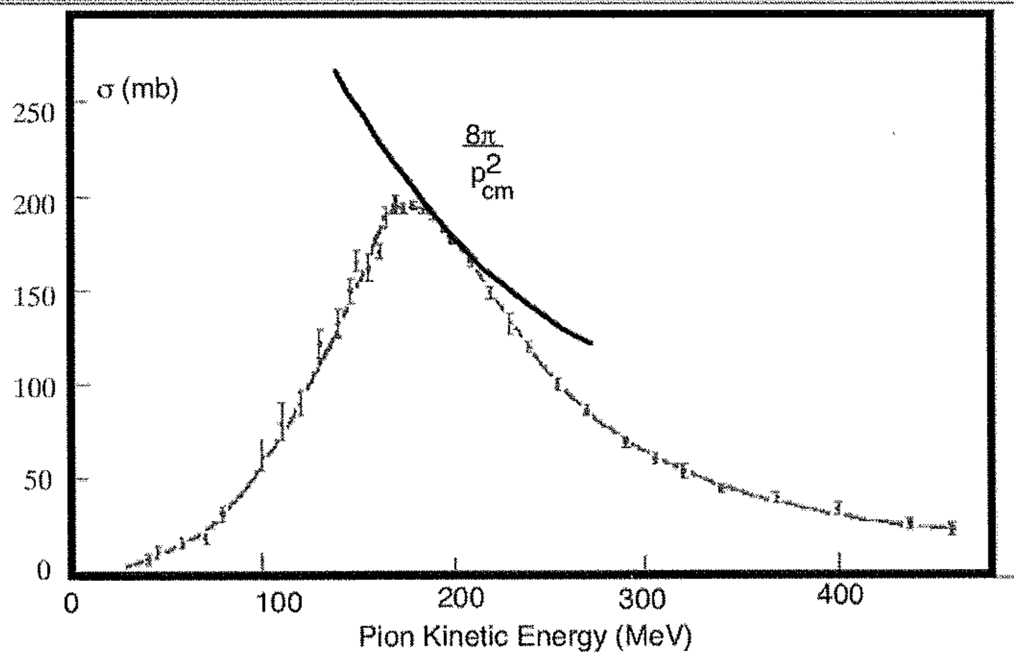


π^+p total cross section: $\Delta^{++}(1236)$



Mass: M = 1232 MeV Full Width: Γ=120 MeV Spin$^{\text{Parity}}$ (J^P): $^3/_2{}^+$

Branching Ratio : $BR(\Delta^{++} \rightarrow \pi^+ p)$=99.5%

$\pi^+p \rightarrow \pi^+p$ cross section at peak of resonance: $\sigma_{peak} = \dfrac{4\pi(\hbar c)^2(2J_{\Delta^{++}}+1)}{p_{cm}^2(2s_{\pi^+}+1)(2s_p+1)} = \dfrac{8\pi(\hbar c)^2}{p_{cm}^2}$