WKB (Wentzel, Kramers, Brillouin) Diff eq $\underbrace{-\psi'' = k^2(x)\psi}$

Like $\ddot{\psi} = -\omega^2\psi$ but $\omega \neq$ const

Note: $\omega = \frac{2\pi}{T}$, $k = 2\pi/\lambda$

For this oscillating ψ, try $\psi \sim e^{iS}$

$$\psi' = iS'\psi$$

$$\psi'' = -S'^2\psi + iS''\psi$$

$$(S'^2 - iS'')\psi = k^2\psi$$

assume $|S''| \ll S'^2$

$\therefore S' = k$ & $S = \int^x k(x)\,dx$ [like kx for const λ case]

iterate: $S = \int^x k(x)\,dx + \varepsilon$ ← assumed "small"

$$S' = k(x) + \varepsilon' \rightarrow S'^2 \approx k^2 + 2k\varepsilon' \quad \text{(stop here)}$$

$$S'' = \underbrace{k'(x)}_{\text{already small}} + \varepsilon'' \leftarrow \text{smaller still}$$

$$\Rightarrow (k^2 + \underbrace{2k\varepsilon' - ik'}) \psi = k^2\psi$$

↳ better be zero: $\varepsilon' = \frac{ik'}{2k} = i\frac{1}{2}(\ln k)' = i(\ln\sqrt{k})'$

$$\Rightarrow \psi = e^{i\int^x k(x)dx - (\ln\sqrt{k} + \text{const})} \sim \frac{1}{\sqrt{k}} e^{i\int k(x)dx}$$

Note: Result allows for approx const $J = \frac{\hbar}{2mi}(\psi^* \overleftrightarrow{\partial} \psi)$

$\int k(x)dx$ is as close as one could get to $k \cdot x$ for non-constant k

$$k = \frac{\sqrt{2m}}{\hbar}(E - V(x))^{1/2} \text{ "=" } \frac{p(x)}{\hbar}$$

So where particle moving fast, k large, $\frac{1}{\sqrt{k}}$ small

$$k = \frac{2\pi}{\lambda} = \frac{p(x)}{\hbar} \longrightarrow \lambda = \frac{h}{p} \leftarrow \text{deBroglie}$$

wave is progressive — no sign of reflection

Condition $k' \ll k^2 \rightarrow \frac{\lambda'}{\lambda^2} \ll \frac{1}{\lambda^2}$ ie $\lambda' \ll 1$

or $\frac{\Delta\lambda}{\lambda} = \frac{\lambda'\lambda}{\lambda} \ll 1$ ie fractional change in wavelength over one wavelength is small

Will ~~#~~ violate if $k = 0$ [turning point]

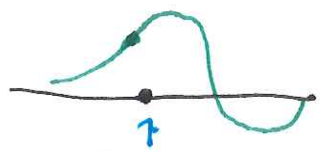
→ For bound states (ie bounded classically allowed with turning points at each end)

Require integer ~~#~~ half wavelengths between nodes

$$\int_{tp}^{tp} k(x)\,dx = \int \frac{2\pi}{\lambda}\,dx = n\pi \leftarrow \infty \text{ square well ends}$$

Long story short (linear potential, Airy function, patch)

for "linear turning points) wave hits turning point at 45° phase → ie $\frac{1}{4}$ of half-wavelength



turning point

$$n\pi \to (n - \tfrac{1}{4} \times \#\text{"linear tp"})\,\pi$$

usually $(n-\frac{1}{2})\pi$

[also written $n+\frac{1}{2}$: choice to start $n=1$ or $n=0$]

Note: the WKB integral $\int \frac{p(x)}{\hbar}\,dx$ is a function of E

ie gives $fn(E) = n$ whereas seek $E = fn(n)$

$$\frac{dp}{dE} = \frac{1}{2}\,\frac{1}{p}\cdot 2m = \frac{m}{p} = \frac{1}{\text{velocity}}$$

$$\text{so } \hbar\pi \frac{dn}{dE} = \int \frac{dx}{\text{velocity}} = \frac{\text{Period}}{2}$$

$$\frac{2\pi}{\text{Period}} = \text{classical } \omega = \frac{1}{\hbar}\frac{dE}{dn}$$

ie — level spacing set by classical frequency

Integrals like $\int p\,dx$ - called action integrals - play a big role in advanced classical mechanics & Bohrs old quantum theory. The fact that sometimes they needed to be quantized as $n\hbar$ & sometimes as $(n+1/2)\hbar$ was a source of confusion.

Eg. H-atom bound state: $E<0$; $E = -\frac{1}{2}\frac{mc^2\alpha^2}{n^2}$

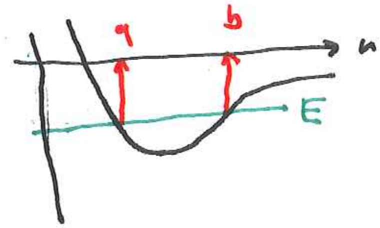
$n = n_r + \ell + 1$

$$-\frac{\hbar^2}{2m}u'' + \frac{\hbar^2\ell(\ell+1)}{2mr^2} - \frac{Ze^2}{4\pi\varepsilon_0 r}u = -|E|u$$

Long story short: $u = Rr$ is zero at $r=0$ not $r=-\infty$ as required by linear patch theory. Make a change of variables to put $r=0$ at $q=-\infty$ ($q = \ln r$); modify diff eq ... $\ell(\ell+1) \longrightarrow (\ell+1/2)^2$ (note they differ by $1/4$)

$$-u'' = \frac{2m}{\hbar^2}\left(-|E| - V(r)\right)u = \frac{2m|E|}{\hbar^2}\left(-1 - \frac{V}{|E|}\right)u$$

$$-1 - \frac{V}{|E|} = \frac{Ze^2}{4\pi\varepsilon_0|E|r} - \frac{\hbar^2(\ell+1/2)^2}{2m|E|r^2} - 1$$



$$= \frac{1}{r^2}\left(\frac{Ze^2}{4\pi\varepsilon_0|E|}r - \frac{\hbar^2(\ell+1/2)^2}{2m|E|} - r^2\right)$$

↳ $-(r-a)(r-b)$ with $ab = \frac{\hbar^2(\ell+1/2)^2}{2m|E|}$

$a+b = \frac{Ze^2}{4\pi\varepsilon_0|E|}$

Note: $\int_a^b \frac{1}{x}\sqrt{-(x-a)(x-b)} = \frac{\pi}{2}\left(\sqrt{b}-\sqrt{a}\right)^2$

$= \frac{\pi}{2}\left(a+b-2\sqrt{ab}\right)$

$$k(r) = \sqrt{\frac{2m|E|}{\hbar^2}}\,\frac{1}{r}\sqrt{-(r-a)(r-b)}$$

$$\int k(x)\,dx = \sqrt{\frac{2m|E|}{\hbar^2}}\,\frac{\pi}{2}\left(\frac{Ze^2}{4\pi\varepsilon_0|E|} - 2\frac{\hbar(\ell+1/2)}{\sqrt{2m|E|}}\right) = \pi\left(n_r + 1/2\right)$$

$$\sqrt{2m}\;Z\overbrace{\frac{e^2}{4\pi\varepsilon_0\hbar c}}^{\alpha}c\;\frac{1}{2}\,\frac{1}{\sqrt{|E|}} = \left(n_r + 1/2 + \ell + 1/2\right)$$

↳ $n_r + \ell + 1 \equiv n$

$$\sqrt{\frac{mc^2}{2}}\,Z\alpha\,\frac{1}{n} = \sqrt{|E|}$$

$$\frac{mc^2\alpha^2 Z^2}{2n^2} = |E| \quad ✓$$