Commutation Relations:

$$[L_i, X_j] = i\hbar \epsilon_{ijk} X_k \qquad \vec{X} = \vec{r}, \vec{p}, \vec{L}$$

$$[L_i, \vec{X}\cdot\vec{Y}] = 0 \qquad \vec{Y} = \text{''}$$

$$p^2 = -\hbar^2 \underbrace{\left(\partial_r^2 + \frac{2}{r}\partial_r\right)}_{\frac{1}{r}\partial_r^2(r\psi)} + \frac{L^2}{r^2}$$ ← we will find eigenfunctions of $L^2 \ni L^2 Y(\theta,\phi) = \hbar^2 \ell(\ell+1) Y$

If we write (separation) $\psi = R(r) Y(\theta, \phi)$

$$\left(\frac{p^2}{2m} + V(r)\right)\psi = E\psi \longrightarrow \left[-\frac{\hbar^2}{2m}\underbrace{\left(\partial_r^2 + \frac{2}{r}\partial_r\right)}_{} + \frac{\hbar^2 \ell(\ell+1)}{2mr^2} + V(r)\right]R = ER$$

if we write $R = \frac{u}{r}$ $\qquad \frac{1}{r}\partial_r^2(rR) = \frac{1}{r}\partial_r^2 u$

$$\rightarrow \left(-\frac{\hbar^2}{2m}\partial_r^2 + \underbrace{\frac{\hbar^2\ell(\ell+1)}{2mr^2}}_{\text{"centrafugal potential"}} + V(r)\right] u = Eu$$ ← looks like 1d

Seeking L^2 eigenfunctions quickly:

$$p^2 = -\hbar^2 \nabla^2 = -\hbar^2\left(\partial_r^2 + \frac{2}{r}\partial_r\right) + \frac{L^2}{r^2}$$

$$\nabla^2 = \partial_r^2 + \frac{2}{r}\partial_r - \frac{\frac{1}{\hbar^2}L^2}{r^2}$$ → dimensionless $L^2 \equiv L'^2$

Consider $(\vec{a}\cdot\vec{r})^\ell = r^\ell f(\theta,\phi)$ where $\vec{a}$ = complex, constant vector

$$\left.\begin{aligned}\partial_x(\vec{a}\cdot\vec{r})^\ell &= \ell(\vec{a}\cdot\vec{r})^{\ell-1} a_x \\ \partial_x^2 &= \ell(\ell-1)(\vec{a}\cdot\vec{r})^{\ell-2} a_x^2\end{aligned}\right\} \Rightarrow \nabla^2(\vec{a}\cdot\vec{r}) = \underset{\substack{\uparrow \\ \text{No Complex Conjugate}}}{a^2}\,\ell(\ell-1)(\vec{a}\cdot\vec{r})^{\ell-2}$$

if $\vec{a} = (1, i, 0) \Rightarrow a^2 = 0$

Note $\vec{a}\cdot\vec{r} = \sin\theta\, e^{i\phi}\, r$

$$0 = \nabla^2(\vec{a}\cdot\vec{r})^\ell = \underbrace{\left(\partial_r^2 + \frac{2}{r}\partial_r\right) r^\ell f}_{[\ell(\ell-1)+2\ell]r^{\ell-2} = \ell(\ell+1)r^{\ell-2}} - \overbrace{\frac{L'^2}{r^2} r^\ell f}^{r^{\ell-2}L'^2 f}$$

So $r^{\ell-2} L'^2 f = \ell(\ell+1) r^{\ell-2} f$

$$L'^2 f = \ell(\ell+1) f$$ ← f is an eigenfunction of L'^2 eigenvalue $\ell(\ell+1)$

$$f = \underbrace{\sin^\ell\theta\; e^{i\ell\phi}}_{\substack{\text{concentrated around equator} \\ \text{all oscillation in } \phi}}$$

Note: Recall from EM $r^\ell P_\ell(\cos\theta)$ solved Laplace

an hence just like previous $L'^2 P_\ell = \ell(\ell+1) P_\ell$

$P_\ell(\cos\theta)$ have no ϕ dependance; all oscillation in θ

Note 2: there are an infinite number of vectors $\ni a^2 = 0$ so we seem to have generated an infinite number of eigenfunctions - But how many are independent?

Find ladder operators which produce a (A) changed eigenvalue of L_z but (B) unchanged eigenvalue of L^2

(A) requires something like $[L_z, Q] = \lambda Q$

(B) requires something like $[L^2, Q] = 0$

Note: $\underset{\downarrow \atop \hbar^2 \ell(\ell+1)}{L^2} = \underbrace{L_x^2 + L_y^2}_{\text{positive definite}} + \underset{\downarrow \atop (\hbar m)^2}{L_z^2}$ } so m cannot get too $\pm$ large $m \sim \pm \ell$

In what follows I'm going to use the dimensionless version of L (but not bother to prime it) as $\hbar$ gets in way

$[L_2, L_3] = i L_1$
$[L_3, L_1] = i L_2$ } consider $[L_3, \underbrace{L_1 \pm i L_2}_{L_\pm}]$

$$[L_3, L_1 \pm i L_2] = [L_3, L_1] \pm i [L_3, L_2]$$
$$= i L_2 \pm L_1 = \pm (L_1 \pm i L_2)$$

Recall: if $[L_z, Q] = \lambda Q$ & $L_z \psi = m \psi$

then $L_z(Q\psi) = (m+\lambda)(Q\psi)$

Pf: $L_z Q \psi - Q L_z \psi = \lambda Q \psi$

$L_z(Q\psi) = m Q\psi + \lambda Q \psi = (m+\lambda) Q\psi$

So $L_\pm$ changes m by ± 1

But m cannot increase without limit. Call the largest possible value of m $\underline{\underline{\ell}}$ then $L_+ \psi_\ell = 0$

Determine the eigenvalue of L^2 ~~the~~ from above!

Note: $L_\pm L_\mp = (L_1 \pm iL_2)(L_1 \mp iL_2)$

$$= L_1^2 + L_2^2 \mp i\overbrace{(L_1L_2 - L_2L_1)}^{iL_3}$$

$$= L_1^2 + L_2^2 \pm L_3$$

so $L^2 = L_1^2 + L_2^2 + L_3^2 = (L_\pm L_\mp \mp L_3) + L_3^2$

Consider $L^2 \psi_\ell = \left[(\underset{\downarrow 0}{L_- L_+} + L_3) + L_3^2\right]\psi_\ell = (\ell + \ell^2)\psi_\ell$

so L^2 eigenvalue is $\ell(\ell+1)$

Lowering the L_3 eigenvalue with L_- does not change L^2 eigenvalue (cuz $[L^2, L_-] = 0$) but must end say at $-k$

$L^2 \psi_{-k} = \ell(\ell+1)\psi_{-k} = [(\overset{0}{L_+ L_-} - L_3) + L_3^2]\psi_{-k}$

$$= (k + k^2)\psi_{-k}$$

so $\ell(\ell+1) = k(k+1) \Rightarrow k = \ell$ or $\underbrace{k = -(\ell+1)}$

but said $m = \ell$ was highest

Now $\ell \to -\ell$ must be reached with integer # steps $\Rightarrow \ell =$ whole # or "half integer"

Let $L_+ |\ell\, m\rangle = A\, |\ell\, m+1\rangle$; find A

$(L_+ |\ell m\rangle)^\dagger L_+ |\ell m\rangle = |A|^2 \langle \ell\, m+1 | \ell\, m+1\rangle$

$\langle \ell m | L_- L_+ | \ell m\rangle = \langle \ell m | L^2 - L_3 - L_3^2 | \ell m\rangle$

$$= \ell(\ell+1) - m(m+1)$$

Similar $L_- |\ell m\rangle = \sqrt{\ell(\ell+1) - m(m-1)}\, |\ell\, m-1\rangle$