beyond 1d: $x \to \vec{r}$, $p_x = \frac{\hbar}{i}\partial_x \to \frac{\hbar}{i}\vec{\nabla}$ $\Big\}$ $H = \frac{p^2}{2m} + V(r) \to \frac{-\hbar^2}{2m}\nabla^2 + V(r)$

Note 1: In EM where we were concerned with Laplace Eq: $\nabla^2 \phi = 0$ we found solutions in spherical coordinates of form $\left(r^\ell \text{ \& } \frac{1}{r^{\ell+1}}\right) L_\ell(\cos\theta)$ because we now seek solutions $\left(\frac{-\hbar^2}{2m}\nabla^2 + V\right)\psi = E\psi$ the radial eqn will become more complex, but the angular part will retain interest. We will need to expand the angular part beyond azimuthal symmetry → ie include ϕ. Thus $L_n(\cos\theta) \to \underbrace{e^{im\phi} L_n^m(\cos\theta)}$ → we will name these Y_n^m

Note 2: In EM polar coordinates we had $(r^m \text{ \& } r^{-m})$ $\begin{matrix}\cos m\phi \\ \sin m\phi\end{matrix}$ Again we'll want to include z & the radial part will become more complex, but the angular part will retain interest which we'll write $e^{im\phi}$

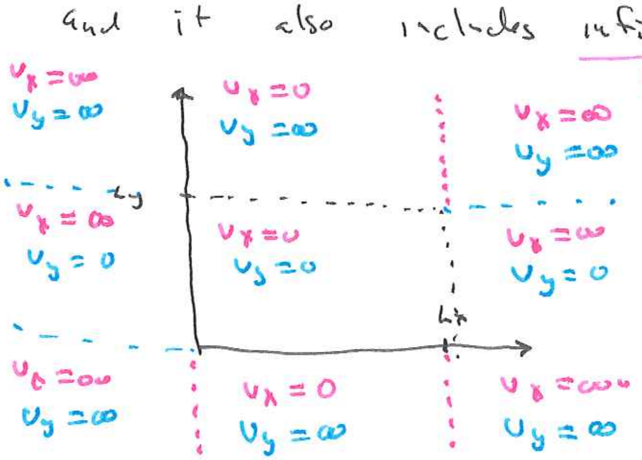
Note 3: In EM rectangular coordinates we had things like $\begin{matrix}\sinh \\ \sin \\ \cos \\ \cosh\end{matrix}$ QM will be a bit easier: we'll only use $\begin{matrix}\sin \\ \cos\end{matrix}$ as $\begin{matrix}\sinh \\ \cosh\end{matrix}$ are exponential & difficult to normalize.

Note 4: In mechanics conservation laws were related to symmetry. Still in classical mechanics, the Poisson Bracket $[H, Q] = 0$ for conservation. In QM the same result for commutators $[H, Q] = 0$ can generate degeneracy as $H\psi = E\psi \Rightarrow H(Q\psi) = E(Q\psi)$

We start with a simple (but uncommon) situation where the PE breaks up: $V(r) = V_x(x) + V_y(y) + V_z(z)$. It does include the 3d isotropic SHO: $V(r) = \frac{1}{2}kr^2 = \frac{1}{2}kx^2 + \frac{1}{2}ky^2 + \frac{1}{2}kz^2$

and it also includes infinite square well: $V_x(x) = \begin{cases} 0 & \text{for } 0 < x < L_x \\ \infty & \text{otherwise} \end{cases}$

└ but not finite



Applies to a free particle

$V_y(y) = \begin{cases} 0 & \text{for } 0 < y < L_y \\ \infty & \text{otherwise} \end{cases}$

This also applies to projectile motion where $U_x = 0$ & $U_y = mgy$ ie no force in x direction, $-mg$ in y

This applies to multiple non interacting particles: $V(\vec{r}_1) + V(\vec{r}_2)$

2d case: $V = V_x(x) + V_y(y) \rightarrow \left(\frac{-\hbar^2}{2m}(\partial_x^2 + \partial_y^2) + V \right)\psi = E\psi$

Let $H_x = \frac{-\hbar^2}{2m}\partial_x^2 + V_x(x)$ & $H_y = \frac{-\hbar^2}{2m}\partial_y^2 + V_y(y)$ then

$(H_x + H_y)\psi = E\psi$. Try separation of variables: $\psi(x,y) = X(x)\,Y(y)$

$Y H_x X + X H_y Y = E(XY) \Rightarrow \frac{1}{X}H_x X + \frac{1}{Y}H_y Y = E \leftarrow$ a constant

so $\frac{1}{X}H_x X = E_x \leftarrow$ some constant & $E = E_x + E_y$

$\frac{1}{Y}H_y Y = E_y \leftarrow$ another

so if X is an energy eigenfunction of H_x: $H_x X = E_x X$

& if Y is an energy eigenfunction of H_y: $H_y Y = E_y Y$

then $\psi = X \cdot Y$ is an energy eigenfunction of the full Hamiltonian with energy $= E_x + E_y$

Example: 2d infinite square well:

$X_{n_x} = \sqrt{\frac{2}{L_x}} \sin\left(\frac{\pi n_x}{L_x}\right) \quad E_x = \frac{\hbar^2 \pi^2 n_x^2}{2m L_x^2}$

$Y_{n_y} = \sqrt{\frac{2}{L_y}} \sin\left(\frac{\pi n_y}{L_y}\right) \quad E_y = \frac{\hbar^2 \pi^2 n_y^2}{2m L_y^2}$

$\psi_{n_x n_y}(x,y) = X_{n_x}(x)\, Y_{n_y}(y) \equiv |n_x n_y\rangle$

Note: I've used orthonormal X & Y which results in orthonormal $\psi_{n_x n_y}$ as

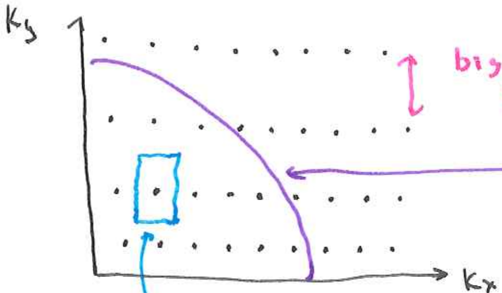
$$\langle n_x' n_y' | n_x n_y \rangle = \int_0^{L_y}\int_0^{L_x} X_{n_x'} Y_{n_y'} X_{n_x} Y_{n_y}\, dx\, dy$$
$$= \int_0^{L_x} X_{n_x'} X_{n_x}\, dx \int_0^{L_y} Y_{n_y'} Y_{n_y}\, dy = \delta_{n_x' n_x}\delta_{n_y' n_y}$$

The states & energies are now labelled by a pair of positive integers: (n_x, n_y). The ground state will be $(1,1)$ & the first excited state will be either $(1,2)$ or $(2,1)$ depending on L_x & L_y : $E = \frac{\hbar^2}{2m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2}\right)$ so if $L_x > L_y$ $E_{21} < E_{12}$

In the interesting special case of a square $(L_x = L_y)$ infinite square well $E_{21} = E_{12} \leftarrow$ degenerate states.

Note that the interesting quantity is really $k_x = \frac{\pi n_x}{L_x}$ & $k_y = \frac{\pi n_y}{L_y}$ as then $E = \frac{\hbar^2}{2m}(k_x^2 + k_y^2)$

Because the states are labelled with a pair of positive integers it is useful to display the states as a lattice of points on a plane → "k" space



bigger spacing in y than x cuz $L_y < L_x$ in this example.

a circle where $k_x^2 + k_y^2$ = constant and hence would have the same energy

a cell of dimension $\frac{\pi}{L_x} \times \frac{\pi}{L_y}$ that includes 1 state
These cells could tile this space.

Estimate the # states with energy < the purple circle by taking the area of that circle over the area of a tile.

→ radius of purple circle $= \sqrt{k_x^2 + k_y^2} = \frac{\sqrt{2mE}}{\hbar}$

→ # states $< E = \dfrac{\frac{1}{4}\pi k^2}{\frac{\pi}{L_x}\frac{\pi}{L_y}} = \frac{1}{4\pi}\frac{2mE}{\hbar^2} L_x L_y$

so $\dfrac{\#\text{states}}{\underbrace{L_x L_y}_{\text{area of well}}} \propto E$

Note 1: we can experimentally make things like this with "2d electron gas" → worth several Nobels in recent decades

Note 2: 3d boxes are easier to make experimentally. The process works much as above except we seek the number of unit cells that fit inside $\frac{1}{8}$ sphere. "fermi sphere"

The result is $\dfrac{\#\text{ states with energy} < E}{\underbrace{L_x L_y L_z}_{\text{volume of well}}} \propto E^{3/2}$

Example 2: 2d isotropic SHO: $V = \frac{1}{2}m\omega^2(x^2+y^2) = \underbrace{\frac{1}{2}m\omega^2x^2}_{V_x(x)} + \underbrace{\frac{1}{2}m\omega^2y^2}_{V_y(y)}$

$$X_{n_x} = \frac{1}{\sqrt{2^n\sqrt{\pi}\,n!}} H_{n_x}(x') e^{-\frac{1}{2}x'^2} \qquad E_x = \hbar\omega(n_x+\tfrac{1}{2}) \qquad E' = 2n+1$$
$$e = \frac{\hbar\omega}{2}$$
$$Y_{n_y} = \frac{1}{\sqrt{2^n\sqrt{\pi}\,n!}} H_{n_y}(y') e^{-\frac{1}{2}y'^2} \qquad E_y = \hbar\omega(n_y+\tfrac{1}{2})$$

Note: n_x & n_y start from zero

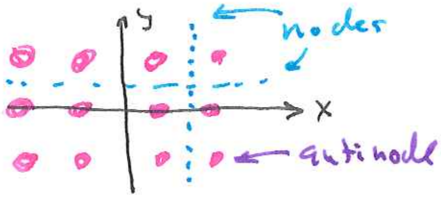
$$|n_x n_y\rangle = \Psi_{n_x n_y}(x,y) = X_{n_x}(x)\, Y(y) \qquad E = \hbar\omega(n_x+n_y+1)$$

Note 1: We used orthonormal X & Y so $\Psi_{n_x n_y}(x,y) = |n_x n_y\rangle$ orthonormal

Note 2: we have degeneracy as $|10\rangle$ & $|01\rangle$ have same energy

Note 3: we have rotational symmetry and hence classically conservation of angular momentum L_z, I therefore expect that if I operate with $L_z = xP_y - yP_x = \frac{\hbar}{i}(x\partial_y - y\partial_x)$ on one of these states I will generate a new & degenerate state.

Note 4: If I were to display $|\Psi|^2$ for one of these states it would look boxy. One would think that if



the problem is rotationally symmetric the solutions would also. To find those rotationally symmetric solutions we must use polar coordinates. But for the time being, let me report that a superposition of these degenerate boxy solutions is rotationally symmetric!!

Note 5: we can use all the raising/lowering operator technology we made before except now there is an operator to raise/lower x: $a^\dagger$ & a And a different operator to raise/lower y: $b^\dagger$ & b

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a) \qquad P_x = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a)$$

$$y = \sqrt{\frac{\hbar}{2m\omega}}(b^\dagger + b) \qquad P_y = i\sqrt{\frac{\hbar m\omega}{2}}(b^\dagger - b)$$

$$H = \hbar\omega(a^\dagger a + b^\dagger b + 1)$$

$$a^\dagger|n_x n_y\rangle = \sqrt{n_x+1}\,|n_x+1\ n_y\rangle$$
$$a\,|n_x n_y\rangle = \sqrt{n_x}\,|n_x-1\ n_y\rangle$$
$$b^\dagger|n_x n_y\rangle = \sqrt{n_y+1}\,|n_x\ n_y+1\rangle$$
$$b\,|n_x n_y\rangle = \sqrt{n_y}\,|n_x\ n_y-1\rangle$$

$$[a,a^\dagger] = [b,b^\dagger] = 1$$
$$0 = [a,b] = [a,b^\dagger] = [a^\dagger,b] = [a^\dagger,b^\dagger]$$

Note 6: It's easy to take this to 3-d $\rightarrow E = \hbar\omega(n_x+n_y+n_z+3/2)$