Seek standing wave solutions to SHO via operators (not diff eq)

But first understand how this process works in the context of diff eq

$$x H_n = \tfrac12 H_{n+1} + n H_{n-1} \Rightarrow x\psi_n = \frac{1}{\sqrt2}\left(\sqrt{n+1}\,\psi_{n+1} + \sqrt{n}\,\psi_{n-1}\right)$$

$$\partial_x H_n = 2n H_{n-1} \Rightarrow \partial_x \psi_n = \frac{1}{\sqrt2}\left(-\sqrt{n+1}\,\psi_{n+1} + \sqrt{n}\,\psi_{n-1}\right)$$

So $\underbrace{\frac{1}{\sqrt2}(x+\partial_x)}_{a \text{ or } a_-}\psi_n = \sqrt{n}\,\psi_{n-1}$ & $\underbrace{\frac{1}{\sqrt2}(x-\partial_x)}_{a^\dagger \text{ or } a_+}\psi_n = \sqrt{n+1}\,\psi_{n+1}$

$x = \frac{1}{\sqrt2}(a+a^\dagger)$ $\quad \times\sqrt{\frac{\hbar}{m\omega}}$

$\partial_x = \frac{1}{\sqrt2}(a-a^\dagger)$ $\quad \times\sqrt{\hbar m\omega}$

$\hookrightarrow p = \frac{1}{i}\partial_x = \frac{1}{\sqrt2\, i}(a-a^\dagger)$

note: $\dagger$ ≡ hermitian conjugate

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start with commutators ≡ $[A,B] = AB - BA$

$$[a,a^\dagger]f = \frac12\Big\{\underbrace{(x+\partial_x)(x-\partial_x)f}_{x^2-\partial_x^2+\underbrace{\partial_x(xf)-x\partial_x f}_{f}} - \underbrace{(x-\partial_x)(x+\partial_x)f}_{x^2-\partial_x^2\underbrace{-\partial_x(xf)+x\partial_x f}_{-f}}\Big\} = f$$

$$[\overset{-\partial_x^2+x^2}{H}, a]f = \frac{1}{\sqrt2}\left\{(-\partial_x^2+x^2)(x+\partial_x)f - (x+\partial_x)(-\partial_x^2+x^2)f\right\} = -2af$$

$$[H,a^\dagger]f = +2a^\dagger f$$

Trick: if $H\psi = E\psi$ then $\begin{cases} H(a^\dagger\psi) = (E+2)(a^\dagger\psi) \\ H(a\psi) = (E-2)(a\psi) \end{cases}$

eigenfunction eigenenergy

there must be a lowest E state (as $E>0$) so

$$a\psi_0 = 0 \rightarrow (x+\partial_x)\psi = 0 \rightarrow \psi \propto e^{-x^2/2}$$

See from above $H = 2a^\dagger a + 1 = 2aa^\dagger - 1$

We have shown: $a^\dagger\psi_n \propto \psi_{n+1}$; find prop constant

Let $a^\dagger\psi_n = c_n\psi_{n+1}$ where c_n to be determined

$$|c_n|^2 = \langle a^\dagger\psi_n | a^\dagger\psi_n\rangle = \langle\psi_n | aa^\dagger\psi_n\rangle = \langle\psi_n \left(\tfrac{H+1}{2}\right)\psi_n\rangle$$
$$= n+1$$

Let $a\psi_n = d_n\psi_{n-1}$ where d_n to be determined

$$|d_n|^2 = \langle a\psi_n | a\psi_n\rangle = \langle\psi_n | a^\dagger a\,\psi_n\rangle = \langle\psi_n | \tfrac{H-1}{2} | \psi_n\rangle$$
$$= n$$