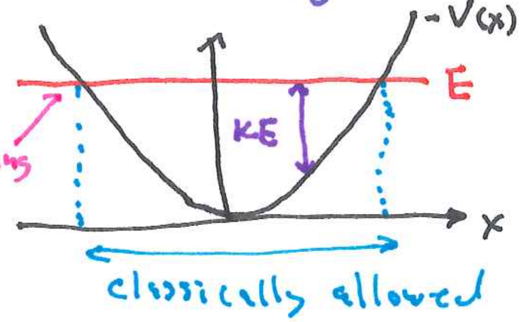
Seek standing wave solutions to SHO via diff eq
(general solution via superposition)



TISE: $-\underbrace{\frac{\hbar^2}{2m}}_{\equiv A}\psi'' + \underbrace{\frac{1}{2}m\omega^2}_{\equiv B}\psi = E\psi$

The following steps are often used to solve this sort of problem

① Go to dimensionless coordinates $\quad x = x'\ell$ ← length unit, $\quad E = E'e$ ← energy unit

$[A] = E\cdot L^2 \qquad [B] = \frac{E}{L^2} \quad \Rightarrow \quad \ell = \left(\frac{A}{B}\right)^{1/4} \qquad e = (A\cdot B)^{1/2}$

$\frac{d}{dx} = \frac{dx'}{dx}\frac{d}{dx'} = \frac{1}{e}\frac{d}{dx'} \qquad \frac{d^2}{dx^2} = \frac{1}{e^2}\frac{d^2}{dx'^2}$

$$-\underbrace{A\frac{1}{\ell^2}}_{=(AB)^{1/2}}\frac{d^2\psi}{dx'^2} + \underbrace{B\ell^2}_{=(AB)^{1/2}} x'^2\psi = E'\underbrace{e}_{(AB)^{1/2}}\psi$$

$$-\frac{d^2\psi}{dx'^2} + x'^2\psi = E'\psi$$ [from now on drop primes]

② Find behavior of ψ as $x \to \infty$: $\underbrace{-\psi'' + x^2\psi}_{\text{must approx cancel}} = E\psi$ ← constant

Factor out that behavior: $\psi = He^{-x^2/2}$ $\quad \Rightarrow \psi = e^{-x^2/2}$

③ Find diff eq for H:

$\psi = He^{-x^2/2}$

$\psi' = H'e^{-x^2/2} - xHe^{-x^2/2}$

$\psi'' = H''e^{-x^2/2} - xH'e^{-x^2/2} - He^{-x^2/2} - xH'e^{-x^2/2} + x^2He^{-x^2/2}$

$= [H'' - 2xH' - H + x^2H]e^{-x^2/2}$

$\Rightarrow -[H'' - 2xH' - H + x^2H] + x^2H = EH$

$-H'' + 2xH' + (1-E)H = 0$

④ Try $H = \sum a_k x^k$

$H' = \sum k a_k x^{k-1}$

$2xH' = \sum 2k a_k x^k$

$H'' = \sum k(k-1)a_k x^{k-2} = \sum (k+2)(k+1)a_{k+2}x^k$

$$\sum\left\{-(k+2)(k+1)a_{k+2} + 2ka_k + (1-E)a_k\right\}x^k = 0$$

$$\frac{2k + (1-E)}{(k+2)(k+1)}a_k = a_{k+2}$$ ← 2 term recursion relation

$a_0 \to a_2 \to a_4 \to a_6$ etc

Note: even k separated from odd a_k

: stop requires $E = 2N+1$ ← N some integer eg if $N=4$, $E=9$; wolog $a_0=1$

$k=0$: $\frac{0+(1-9)}{2}\cdot 1 = -4 = a_2$

$k=2$ $\frac{4+(1-9)}{4\cdot 3}(-4) = \frac{4}{3} = a_4$ ← all higher a_k are zero

$\Big\} \; H = 1 - 4x^2 + \frac{4}{3}x^4$

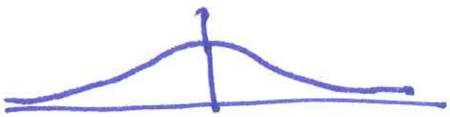
we have found a solution of form

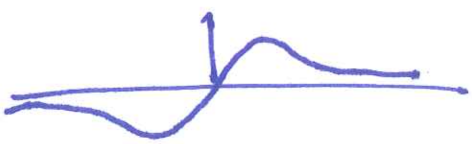
$$\psi_n = N_n \, H_n(x) \, e^{-x^2/2} \quad \rightarrow \quad \text{"}x\text{"} = x\sqrt{\frac{m\omega}{\hbar}}$$

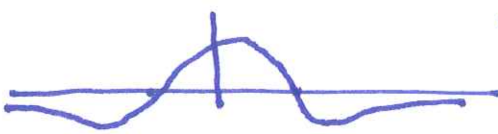
$$E_n = 2n+1 \qquad\qquad E_n = (n+1/2)\hbar\omega$$

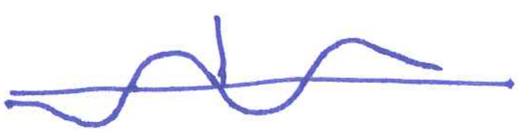
Hermite poly

to be found normalization

ψ_0:  $\sim e^{-x^2/2}$ even

ψ_1:  $\sim x e^{-x^2/2}$ odd

ψ_2:  $\sim (2x^2-1)e^{-x^2/2}$ even

ψ_3  $\sim (2x^3-3x)e^{-x^2/2}$ odd

These functions are <u>complete</u> (ie can write any function as a superposition of those)

$$f(x) = \sum c_n \psi_n$$

$c_n \leftarrow \langle \psi_n | f \rangle$

Note: most often wavefunctions are displayed as a up-shifted stack along with the PE.
The zero line for each ψ is the energy of that ψ

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