The potential

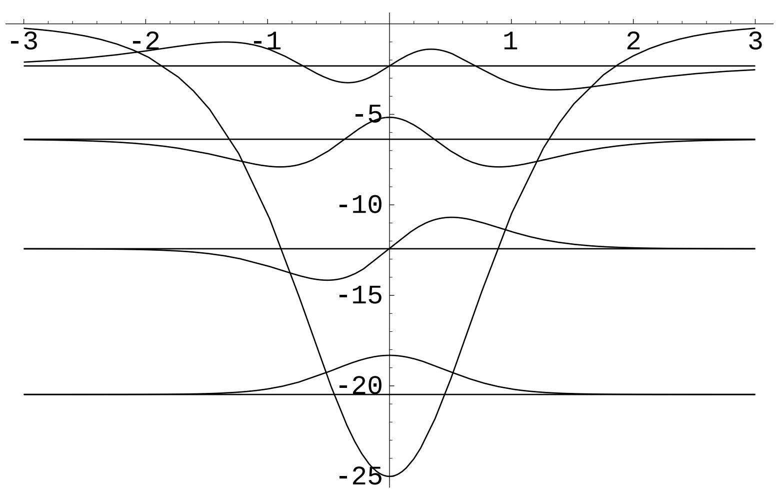
$$V(x) = -\frac{V_0}{\cosh^2(x/a)}$$

is exactly solvable and so provides a test case for various approximation methods. First as usual go to dimensionless coordinates with unit length $l = a$ and unit energy $e = \hbar^2/(2ma^2)$:

$$\begin{aligned}
\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi - \frac{V_0}{\cosh^2(x/a)}\psi &= E\psi \\
\frac{-\hbar^2}{2ma^2}\frac{\partial^2}{\partial x'^2}\psi - \frac{V_0}{\cosh^2(x')}\psi &= E\psi \\
-\frac{\partial^2}{\partial x'^2}\psi - \frac{V_0'}{\cosh^2(x')}\psi &= E'\psi
\end{aligned}$$

(As usual on the following pages we will simplify by not writing the primes.) Note that the potential resembles a finite square well in that as $|x| \to \infty$ the potential approaches zero. There are only a finite number of bound states ($E < 0$) in addition to the continuum of free ($E > 0$) states. Note that as $|x| \to 0$ the potential looks quadratic, and so the low-energy solutions should look like SHO solutions (e.g., in having equally spaced eigenenergies).

Here is a stacked-wavefunction plot showing the four lowest states for $V_0' = 25$:



The exact eigenenergies are given by:

$$E_n' = -\left[\sqrt{V_0' + \tfrac{1}{4}} - (n + \tfrac{1}{2})\right]^2$$

for $n = 0$ up to the maximum value of n for which the value in square brackets ([ ]) is positive.

1. Use the Rayleigh-Ritz (variational) method to estimate the eigenenergy of the ground state and first excited state for $V_0 = 25$. Use the trial wavefunctions:

$$\begin{aligned} \psi_0 &= \cosh(x)\exp(-qx^2) \\ \psi_1 &= x\cosh(x)\exp(-qx^2) \end{aligned}$$

Why are these reasonable choices? (Words please!) You will need to use *Mathematica* to do the integrals. Since you have symmetry, you might as well integrate only over the range $[0, \infty]$:

```
$Assumptions=$Assumptions && {q>1}
f[x_]=Cosh[x] Exp[- q x^2]
ke=Integrate[f'[x]^2,{x,0,Infinity}]
pe=-25 Integrate[f[x]^2/Cosh[x]^2,{x,0,Infinity}]
n=Integrate[f[x]^2,{x,0,Infinity}]
e=(ke+pe)/n
```

(The first line is to convince *Mathematica* that the integral actually converges... if $q < 0$ the integrand would blow up as $|x| \to \infty$.) Use the *Mathematica* function `FindMinimum` to do the minimization:
`FindMinimum[e,{q,`*your guess here*`}]`

Note that you must give *Mathematica* a starting guess for q. I'd plot E vs. q to find a good guess for the minimum.

Compare your estimates to the exact eigenenergy given above. Plot both normalized wavefunctions using code similar to:
`Plot[Evaluate[f[x]/Sqrt[n] /. q->`*your result here*`],{x,-2,2}]`

2. Use the Rayleigh-Ritz (variational) method to estimate the eigenenergy of the ground state and first excited state for $V_0 = 25$. Use the trial wavefunctions:

$$\begin{aligned} \psi_0 &= 1/\cosh^q(x) \\ \psi_1 &= \sinh(x)/\cosh^q(x) \end{aligned}$$

Why are these reasonable choices? (Words please!) These integrals are simple enough that pencil and paper may be easier than *Mathematica*. *Mathematica* will express some results in terms of the hypergeometric ${}_2F_1$ whereas using the below, you can do better:

$$\int_0^\infty \cosh^{-2q}(x)\,dx = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(q)}{\Gamma(q+\frac{1}{2})}$$

So:

$$\frac{\langle\psi_0|\cosh^{-2}|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle} = \frac{\Gamma(q+1)/\Gamma(q+\frac{3}{2})}{\Gamma(q)/\Gamma(q+\frac{1}{2})} = \frac{q}{q+\frac{1}{2}} \qquad \text{Since: } \Gamma(x+1) = x\Gamma(x) = x!$$

and

$$\sinh^2(x) = \cosh^2(x) - 1$$

Compare your estimates to the exact eigenenergy given above. What can you conclude? Plot both normalized wavefunctions as in #1.

3. Since the potential looks quadratic for $x \sim 0$, we should be able to approximate using SHO. Thus since:

$$\cosh^{-2}(y) = 1 - y^2 + \frac{2}{3}y^4 - \frac{17}{45}y^6 + \cdots$$

we have:

$$\begin{aligned}
\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi - \frac{V_0}{\cosh^2(x/a)}\psi &= E\psi \\
\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x'^2}\psi + \frac{V_0}{a^2}x^2\psi - \frac{V_0 2}{3a^4}x^4\psi &\approx (E+V_0)\psi \\
-A\frac{\partial^2}{\partial x^2}\psi + B\,x^2\psi - B\frac{2}{3a^2}x^4\psi &\approx (E+V_0)\psi
\end{aligned}$$

Using the length scale: $l = (A/B)^{\frac{1}{4}}$ and energy scale: $e = (AB)^{\frac{1}{2}}$, we have:

$$\begin{aligned}
\frac{-A}{(A/B)^{\frac{1}{2}}}\frac{\partial^2}{\partial x'^2}\psi + B(A/B)^{\frac{1}{2}}x'^2\psi - B(A/B)^{\frac{1}{2}}\frac{2l^2}{3a^2}x'^4\psi &\approx (E+V_0)\psi \\
-\frac{\partial^2}{\partial x'^2}\psi + x'^2\psi - \frac{2l^2}{3a^2}x'^4\psi &\approx (E'+V_0')\psi \\
H_0\psi - \frac{2l^2}{3a^2}x'^4\psi &\approx (E'+V_0')\psi
\end{aligned}$$

where H_0 is the SHO Hamiltonian with eigenenergies $E_n'^{(0)} = (2n+1)$. Thus $E' + V_0' = (E+V_0)/e = 2n+1$ or

$$E = -V_0 + \left(\frac{\hbar^2}{2ma^2}V_0\right)^{\frac{1}{2}}(2n+1)$$

Dividing through by $\hbar^2/2ma^2$ to produce the dimensionless quantities introduced for the $1/\cosh^2$ potential, we have:

$$E' = -V_0' + (V_0')^{\frac{1}{2}}(2n+1)$$

Whereas the exact answer is:

$$E' = -\left[\sqrt{V_0' + \tfrac{1}{4}} - (n+\tfrac{1}{2})\right]^2 = -(V_0' + \tfrac{1}{4}) + \left(V_0' + \tfrac{1}{4}\right)^{\frac{1}{2}}(2n+1) - (n+\tfrac{1}{2})^2$$

Use first order perturbation theory to find how the x^4 term affects the eigenenergies. Use second order perturbation theory to find the effect on the ground state. Hint: remember your raising and lower operators!

4. If we make the $1/\cosh^2$ potential very deep and very narrow then it should approximate a delta function potential. Since:

$$\begin{aligned} \int_0^\infty V(x)\,dx &= -\int_0^\infty V_0/\cosh^2(x/a)\,dx = -V_0 a \int_0^\infty 1/\cosh^2(y)\,dy \\ &= -V_0 a \int_0^\infty \tanh'(y)\,dy = -V_0 a \end{aligned}$$

we can keep the delta function potential strength $w = 2V_0 a$ a constant as $V_0 \to \infty$ and $a \to 0$. See if the limit of the $1/\cosh^2$ potential ground-state eigenenergy agrees with the delta function potential results derived on the web. What happens to the other bound states? (Note: in the above equations V_0 is the actual potential not the dimensionless V_0')

5. Find the WKB approximation for these eigenenergies. Hint: change variables in the WKB integral to $u = \sinh(x)$, note closely the range of integration in u and use the fact:

$$\int_0^A \frac{\sqrt{A^2 - u^2}}{1 + u^2}\,du = \frac{\pi}{2}\left(\sqrt{1 + A^2} - 1\right)$$

P.S. For folks knowing contour integration: Prove the above integral for extra credit.