Physics 346: QM EXAM 2 28 October 2013

1. The following state is a normalized eigenfunction of $\vec{J}^2$, J_z, $\vec{L}^2$, and $\vec{S}^2$, where $\vec{J} = \vec{L} + \vec{S}$.

$$\psi = \sqrt{\tfrac{4}{7}} Y_{31}\chi_- - \sqrt{\tfrac{3}{7}} Y_{30}\chi_+$$

(Notation: $Y_{\ell m}(\theta, \phi)$ are spherical harmonics; χ_+ is our textbook's notation for a spin up electron which I've often denoted $\uparrow$; χ_- is similarly $\downarrow$.) What are the eigenvalues of J_z, $\vec{L}^2$, and $\vec{S}^2$ for this state? Write out $\vec{J}^2$ in terms of things like $L_\pm, S_\mp, L^2, \ldots$ and then show $\vec{J}^2\ \psi = \hbar^2 j(j+1)\psi$ (i.e., that ψ is an eigenfunction of $\vec{J}^2$) and in the process determine the value of j.

2. Consider the case of a spin 1 particle. The usual basis states for this particle would be eigenfunctions of both S_z (with eigenvalues $\hbar m$ where $m = -1, 0, 1$) and S^2 (all with eigenvalue $\hbar^2 s(s+1)$ where $s = 1$). We denote these basis states as $|sm\rangle$:

$$|1\ -1\rangle = \chi_- = \downarrow \qquad |1\ 0\rangle = \chi_0 = \rightarrow \qquad |1\ +1\rangle = \chi_+ = \uparrow$$

Using raising and lowering operators, show what happens when the operator S_x operates each of these basis states. Consider then a general state: $\psi = a\chi_+ + b\chi_0 + c\chi_-$ (equivalent to the column vector: (a, b, c)). The result of S_x operating on such a general state and be expressed as a matrix, i.e.,

$$\begin{bmatrix} & & \\ & S_x & \\ & & \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Leftrightarrow S_x\psi$$

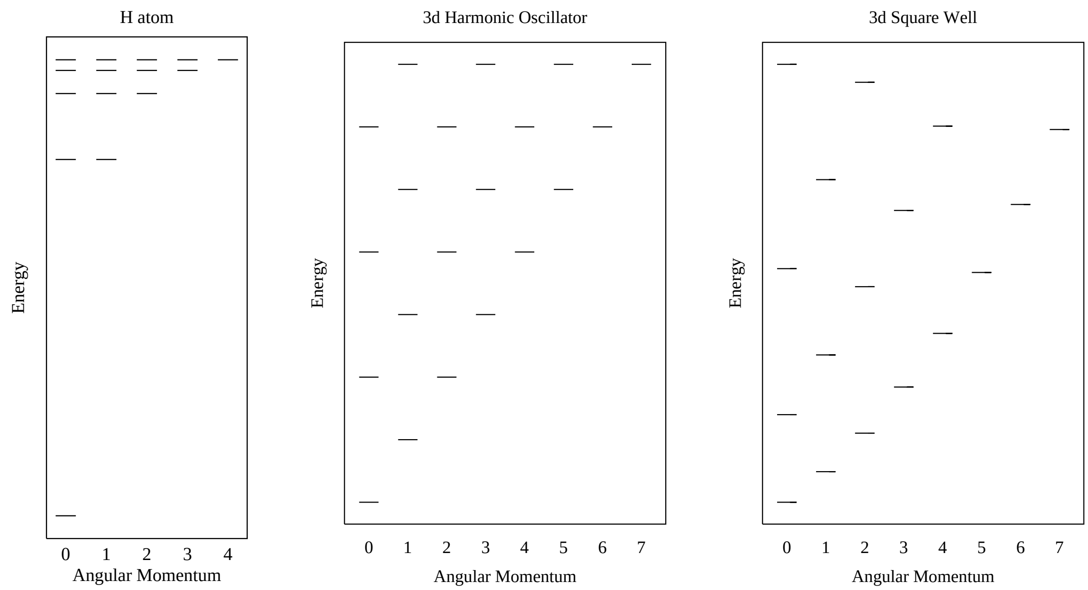
Display the resulting matrix form of S_x.

3. Recall the particle-in-a-box problem with $V(x) = 0$ for $0 < x < L$, but $V(x) = \infty$ elsewhere with orthonormal eigenfunctions ψ_n $(n = 1, 2, 3, \ldots)$:

$$E_n = \frac{(q\hbar)^2}{2m} \qquad \psi_n(x) = \sqrt{\frac{2}{L}}\ \sin(qx) \qquad \text{where: } q = \frac{n\pi}{L}$$

Consider two identical, but noninteracting, electrons in this box. Write down the lowest energy wavefunction for the two electrons if they are in a singlet state. (Feel free to use a compact notation like $\psi_2(x_1)$ instead $\sqrt{\frac{2}{L}}\ \sin(2\pi x_1/L)$.) Write down the lowest energy wavefunction for the two electrons if they are in a triplet state. Consider three identical, but noninteracting, spin-0 particles in this box. Write down the lowest energy wavefunction for the three particles.

4. The below energy level diagrams show the ground state and several excited states for the H-atom, 3-d simple harmonic oscillator, and 3-d infinite square well.



(a) Notice that in the H-atom the splitting between neighboring states with the same angular momentum decreases with increasing energy whereas in the infinite square well that same splitting increases with increasing energy, and in the SHO that spacing is uniform. How does this QM phenomenon relate to the classical mechanics of the same system? Sketch the classical trajectory (orbit) in the case of small angular momentum for an electron in an H-atom and also for an electron in an infinite square well.

(b) In each of the above three diagrams, circle the state that could be called "$3d$". Define 'degeneracy'. What is the degeneracy of a $3d$ state?

(c) Consider the entire set of $n = 3$ H-atom states. What is the total degeneracy of these states? In the case of $Z > 1$ atoms, we have the phenomena of "ℓ-tilting". Sketch the energy level diagram of just the $n = 3$ states for a $Z > 1$ atom thus displaying the meaning of "ℓ-tilting".

(d) Keeping in mind the phenomena of "ℓ-tilting" and using the notation $\uparrow$ to denote a $S_z = \hbar/2$ electron and $\downarrow$ to denote a $S_z = -\hbar/2$ electron, display (writing the arrows directly on the above energy level diagram for H) how the 19 electrons of Potassium would occupy the above states. Write down the electron configuration of Potassium using the standard notation.

(e) In its ground state Carbon has two p electrons. What are the possible values for total orbital angular momentum (L) and total spin angular momentum (S) and total total angular momentum (J) keeping in mind that electrons are fermions. Display these possible states in proper spectroscopic notation and report which state Hund's Rules claim is the ground state of Carbon.