Complete no more than 5 of the following problems

1. (a) Given a wavefunction $\psi(x)$ explain how you could calculate the probability that measurement of some quantity Q would find a particular value q. How does one determine the values q that are actually possible?

   (b) Given a wavefunction $\psi(x)$ explain how you could calculate the probability the particle would be found in the interval $a \le x \le b$.

2. Consider the harmonic oscillator (with, as usual, spring constant: $k = m\omega^2$) in one dimension with normalized energy eigenfunctions $u_n(x)$ of energy $E_n = \hbar\omega(n + \frac{1}{2})$. Normalize the wave function
$$\psi(x) = 5u_1(x) + (1 + i)u_2(x) + 3u_3(x)$$
Find the probability the particle would be found in the $n = 2$ state.[1] Calculate $\langle\psi|H|\psi\rangle$. Write down an expression for $\psi(x, t)$ (i.e., how will ψ change in the future).

3. In the notation of the previous problem, consider the state:
$$\psi(x, t) = \left(u_0(x)e^{-i\omega t/2} + iu_1(x)e^{-3i\omega t/2}\right)/\sqrt{2}$$
Using raising and lowering operators, directly calculate: $\langle\psi(x, t)|x|\psi(x, t)\rangle$ and $\langle\psi(x, t)|p|\psi(x, t)\rangle$. Show that your answers satisfy the equation:
$$\frac{d}{dt}\langle\psi(x, t)|x|\psi(x, t)\rangle = \frac{\langle\psi(x, t)|p|\psi(x, t)\rangle}{m}$$
Briefly describe why both $\langle u_n(x)|x|u_n(x)\rangle$ and $\langle u_n(x)|p|u_n(x)\rangle$ are zero for every n and why that is consistent with what one would expect in the context of classical mechanics.

4. Consider a particle in a box with $V(x) = 0$ for $0 < x < L$ and $V(x) = +\infty$ elsewhere. The normalized energy eigenfunctions for this problem are: $u_n(x) = \sqrt{\frac{2}{L}}\sin(kx)$ where $k = n\pi/L$ and the corresponding energy eigenvalue is $E_n = (\hbar k)^2/(2m)$. At $t = 0$ the particle has wavefunction
$$\psi(x) = \begin{cases} N & \text{if } 0 < x < L/2 \\ 0 & \text{elsewhere} \end{cases}$$
Find the value of N needed to normalize this ψ. What is the probability that the particle would be found in the ground state (i.e., $n = 1$)? The future of this state (i.e., $\psi(x, t)$ valid for $t \ge 0$) can be expressed as a linear combination of stationary states. Write down this infinite series expression for $\psi(x, t)$ and report how you could find the expansion coeficients.

[1] Griffiths would prefer I used the longer phrase: "Find the probability that a measurement of the particle's energy produces a value of E_2". As you know physicists have a great love for concise expressions which may result in ambiguity if you don't know how to decode them. However, you do need to know how to decode the common expressions of QM whether Griffiths approves of them or not.

5. Prove the following formulae for commutators. In the below $f(x)$ is an arbitrary function and p is the momentum operator.

   (a) $[p,x]f(x) = \dfrac{\hbar}{i}\, f(x)$

   (b) $[V(x),p]f(x) = -\dfrac{\hbar}{i}\dfrac{\partial V}{\partial x}\, f(x)$

   (c) $[p^2,x]f(x) = 2\,\dfrac{\hbar}{i}\, p\, f(x)$

   (d) $[H,x] = \dfrac{\hbar}{im}\, p$ where $H = \dfrac{p^2}{2m} + V(x)$

   (e) $[A,BC] = [A,B]C + B[A,C]$

6. According to *Mathematica* the tridiagonal matrix:

$$\mathcal{M} = \begin{pmatrix} a & 2b & 0 & 0 \\ 2b & a & 2b & 0 \\ 0 & 2b & a & 2b \\ 0 & 0 & 2b & a \end{pmatrix}$$

   has eigensystem (eigenvector $\leftrightarrow$ eigenvalue, i.e., $\vec{v}_i \leftrightarrow \lambda_i$ where $\mathcal{M}\vec{v}_i = \lambda_i \vec{v}_i$):

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 1+\sqrt{5} \\ 1+\sqrt{5} \\ 2 \end{pmatrix} \leftrightarrow \lambda_1 = a + b\left(1+\sqrt{5}\right) \qquad \vec{v}_2 = \begin{pmatrix} 2 \\ 1-\sqrt{5} \\ -(1-\sqrt{5}) \\ -2 \end{pmatrix} \leftrightarrow \lambda_2 = a + b\left(-1+\sqrt{5}\right)$$

$$\vec{v}_3 = \begin{pmatrix} 2 \\ 1-\sqrt{5} \\ 1-\sqrt{5} \\ 2 \end{pmatrix} \leftrightarrow \lambda_3 = a + b\left(1-\sqrt{5}\right) \qquad \vec{v}_4 = \begin{pmatrix} 2 \\ -(1+\sqrt{5}) \\ 1+\sqrt{5} \\ -2 \end{pmatrix} \leftrightarrow \lambda_4 = a - b\left(1+\sqrt{5}\right)$$

   (a) Directly calculate (by hand) $\mathcal{M}\vec{v}_1$ and show that your result equals $\lambda_1\vec{v}_1$.

   (b) Use your calculator to find N so that $N\vec{v}_1$ is a unit vector. (I.e., normalize $\vec{v}_1$.)

   (c) Select two distinct eigenvectors and show that they are orthogonal.

   (d) I claim that any vector can be expressed as a linear combination of these four eigenvectors. Find the expansion coefficent α_1 for the particular vector displayed below.

$$\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 + \alpha_4\vec{v}_4$$

7. Consider the case of motion through a positive potential bump (rather then the potential well discussed in class):

$$V(x) = \begin{cases} 0 & \text{if } x < 0 \\ +V_0 & \text{if } 0 \le x \le L \\ 0 & \text{if } L < x \end{cases}$$

where the particle energy E is greater than the bump height V_0.

   (a) Draw a graph of the potential $V(x)$. Show on your graph a constant energy $E > V_0 > 0$.

   (b) Write down the time independent Schrödinger equation. Solve that differential equation and record solutions for each of the regions: $x \in (-\infty, 0)$, $x \in (0, L)$, and $x \in (L, +\infty)$.

   (c) You should be able to write your solutions (or linear combinations of your solutions) in the form:

$$\psi(x) = \begin{cases} e^{ikx} + Re^{-ikx} & \text{if } x < 0 \\ A\sin(qx) + B\cos(qx) & \text{if } 0 \le x \le L \\ Te^{ikx} & \text{if } L < x \end{cases}$$

   Report formula for k and q. What is the physical significance of R and T?

   (d) I count four unknowns in this problem... what are they?

   (e) So we need four equations that have to do with connecting ψ between the regions. Write down the four required equations. (You do not nedd to solve these equations.)

Extra Credit: For the situation of #7 above, describe (in words) what Newton would say the motion would be like.