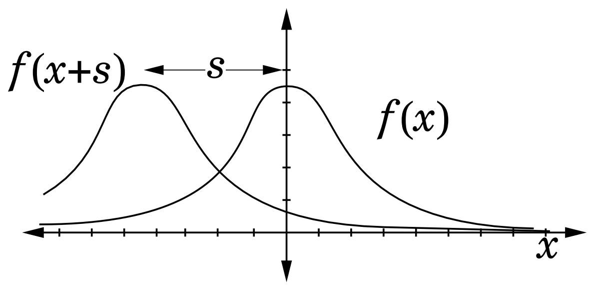
Recall problem 3.39 (p. 129) where you showed the the momentum operator p "generated translation" in the sense that:

$$f(x+s) = e^{isp/\hbar}\, f(x)$$

Graphically, $f(x+s)$ is pushed to the left[1] of $f(x)$: 'translated'. This means, for example, if $f(x)$ has a maximum at $x=0$, $e^{isp/\hbar}\, f(x)$ will have a maximum at $x=-s$.



In an exactly analogous fashion show that L_z generates rotations about the z-axis, in the sense that:

$$f(\phi+s) = e^{isL_z/\hbar}\, f(\phi)$$

Directly calculate the result of $e^{isL_z/\hbar}$ operating on the function $f(\phi)=\phi^2$ and show that the result is $(\phi+s)^2$.

Similarly it would be nice to show that L_y generates rotations about the y-axis, but L_y has a fairly complex form, which makes such a proof difficult. It is simpler (if not exactly simple) to take a particular function and show that it is rotated. The function I want you to work with is

$$\cos\theta = \hat{\mathbf{r}}\cdot\hat{\mathbf{k}}$$

This function takes its maximum on the positive z axis; if rotated it will take its maximum 'before' the z axis, i.e., along the ray defined by the backward rotated $\hat{\mathbf{k}}$ vector:

$$\hat{\mathbf{e}} = (-\sin s, 0, \cos s)$$

so the result of $e^{isL_y/\hbar}$ operating on $\cos\theta$ should be:

$$\hat{\mathbf{r}}\cdot\hat{\mathbf{e}} = (\sin\theta\,\cos\phi, \sin\theta\,\sin\phi, \cos\theta)\cdot(-\sin s, 0, \cos s) = \cos\theta\cos s - \sin\theta\cos\phi\sin s$$

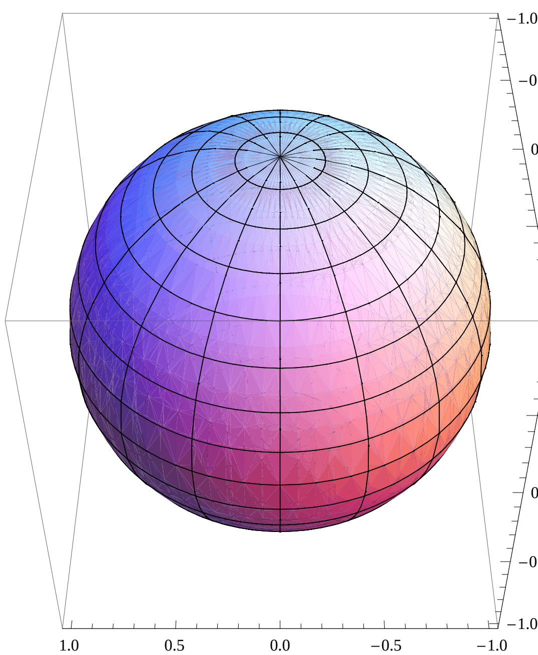
You must therefore calculate:

$$e^{isL_y/\hbar}\cos\theta = \sum_{k=0}^{\infty}\frac{(is/\hbar)^k}{k!}\,L_y^k\cos\theta$$

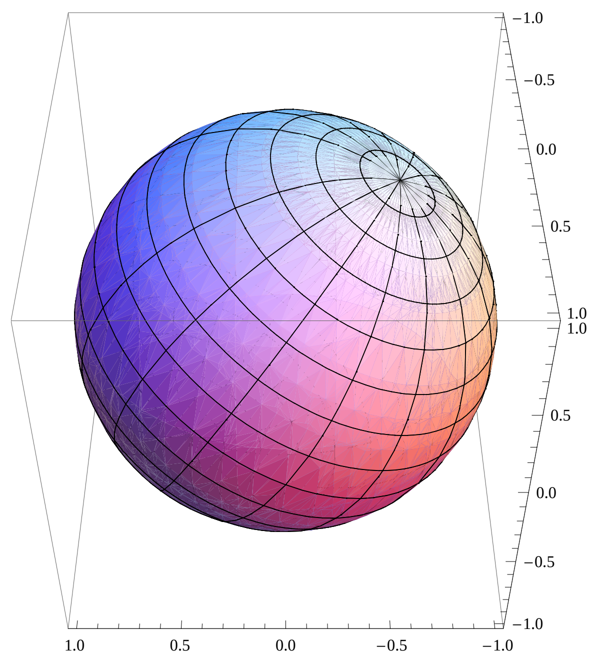
where $L_y^k\cos\theta$ means that L_y operates k times on the function $\cos\theta$, and show that it produces the above result. Calculating an infinite number of derivatives does not sound easy, but a pattern should appear. Do remember the Taylor's expansion for the trigonometric functions:

$$\sin s = s - \frac{s^3}{3!} + \frac{s^5}{5!} - \frac{s^7}{7!} + \ldots \qquad \cos s = 1 - \frac{s^2}{2!} + \frac{s^4}{4!} - \frac{s^6}{6!} + \ldots$$

[1] An equivalent formulation is that the function was unchanged but the origin shifted to the right, so that the maximum previously at $x=0$ now occurs at the coordinate $x=-s$.



(a) The function $\cos\theta$ has its maximum at the top pole.



(b) That maximum (the pole) now occurs at a negative y angle on the back-rotated (tilted) sphere