$$\int_{-\infty}^{+\infty} e^{-\alpha x^2-\beta x}\,dx = \sqrt{\frac{\pi}{\alpha}}\,e^{\beta^2/4\alpha} \qquad \int_{-\infty}^{+\infty} x^2 e^{-\alpha x^2}\,dx = \frac{-d}{d\alpha}\int_{-\infty}^{+\infty} e^{-\alpha x^2}\,dx \qquad \int_0^{\infty} x^n\,e^{-\alpha x}\,dx = n!/\alpha^{n+1} \qquad \int f(x)\delta(x-a)\,dx = f(a)$$

$$H\psi = i\hbar\partial_t\psi \qquad H\psi = E\psi \qquad H = \frac{p^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\,\partial_x^2 + V(x) \qquad p = -i\hbar\partial_x \qquad [p,x] = -i\hbar$$

$$\partial_t\ \psi^*(x,t)\psi(x,t) = -\partial_x J \qquad \text{where current } J = \frac{\hbar}{2im}\left(\psi^*\,\partial_x\psi - \psi\,\partial_x\psi^*\right) = \frac{\hbar}{2im}\ \psi^*\ \overleftrightarrow{\partial_x}\ \psi$$

$$\frac{d}{dt}\langle\psi|A|\psi\rangle = \langle\psi|\partial_t A|\psi\rangle + \langle\psi|i[H,A]/\hbar|\psi\rangle \qquad \sigma_A\sigma_B = \Delta A\,\Delta B \geq \frac{1}{2}\,|\langle\psi|i[A,B]|\psi\rangle|$$

Particle-in-a-box with $V(x) = 0$ for $0 < x < L$, but $V(x) = \infty$ elsewhere

$$E_n = \frac{(\hbar k)^2}{2m} \qquad u_n(x) = \sqrt{\frac{2}{L}}\,\sin(kx) \quad \text{where} \quad k = \frac{n\pi}{L} \qquad n = 1,2,3\ldots$$

3-d: $|n_x n_y n_z\rangle = u_{n_x}(x)u_{n_y}(y)u_{n_z}(z) \quad E = \dfrac{(\hbar k)^2}{2m}$ where $\vec{\mathbf{k}} = (n_x\pi/L_x, n_y\pi/L_y, n_z\pi/L_z)$

fermi ($s=\frac{1}{2}$) gas: $\mathcal{N} = 2\times\ \dfrac{(2mE)^{3/2}}{6\pi^2\hbar^3}\ V \qquad E_F = \dfrac{\hbar^2}{2m}\ \left(3\pi^2 n\right)^{2/3} \qquad P = \dfrac{2}{3}\ \langle E\rangle\ n = \dfrac{2}{5}\ E_F\ n$

Delta Function: $V(x) = -\alpha\delta(x) \Longrightarrow \Delta\psi' = -\dfrac{2m\alpha}{\hbar^2}\ \psi(0) \qquad \psi(x) = \sqrt{\kappa}\,e^{-\kappa|x|}$ where: $\kappa = m\alpha/\hbar^2 \qquad E = -\dfrac{\kappa^2\hbar^2}{2m}$

Harmonic Oscillator with $V(x) = \dfrac{1}{2}\,m\omega^2 x^2 \qquad E_n = \hbar\omega\,(n+\frac{1}{2}) \qquad n = 0,1,2\ldots$

$|n\rangle = u_n(x) = N_n H_n(\xi)\,e^{-\frac{1}{2}\xi^2}$ where $\xi = \sqrt{\dfrac{m\omega}{\hbar}}\ x$ and H_n is an n^{th} degree polynomial

$$a_- = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p}{m\omega}\right) = \frac{1}{\sqrt{2}}\,(\xi + \partial_\xi) \qquad a_+ = a_-^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - i\frac{p}{m\omega}\right) = \frac{1}{\sqrt{2}}\,(\xi - \partial_\xi) \qquad x = \sqrt{\frac{\hbar}{2m\omega}}\ (a_+ + a_-)$$

$$[a_-, a_+] = 1 \qquad [H, a_\pm] = \pm\hbar\omega a_\pm \qquad H = \hbar\omega\left(\tfrac{1}{2} + a_+a_-\right) \qquad a_-\,|n\rangle = \sqrt{n}\ |n-1\rangle \qquad a_+\,|n\rangle = \sqrt{n+1}\ |n+1\rangle$$

3-d: $\left(n_x + n_y + n_z + \frac{3}{2}\right) = \left(2n_r + \ell + \frac{3}{2}\right) \qquad \psi = \sqrt{\dfrac{2\cdot n_r!}{(\ell+\frac{1}{2}+n_r)!}}\ r'^{\ell}\ L_{n_r}^{\ell+\frac{1}{2}}(r'^2)\ e^{-r'^2/2}\ Y_{\ell m}(\theta,\phi)$

Angular Momentum: $\vec{\mathbf{L}} = \vec{\mathbf{r}}\times\vec{\mathbf{p}} \qquad [L_i, V_j] = i\hbar\epsilon_{ijk}V_k$ where vector $\vec{\mathbf{V}} = \vec{\mathbf{r}}, \vec{\mathbf{p}}, \vec{\mathbf{L}} \qquad |\ell m\rangle = Y_{\ell m}(\theta,\phi) \qquad -\ell \leq m \leq +\ell$

$$\vec{\mathbf{L}}^2\,|\ell m\rangle = \ell(\ell+1)\,\hbar^2\,|\ell m\rangle \qquad L_z\,|\ell m\rangle = m\hbar\,|\ell m\rangle \qquad L_\pm\,|\ell m\rangle = \sqrt{\ell(\ell+1) - m(m\pm 1)}\ \hbar\,|\ell\,m\pm 1\rangle$$

$$L_\pm = L_x \pm iL_y \qquad [L_+, L_-] = 2\hbar L_z \qquad [L_z, L_\pm] = \pm\hbar L_\pm \qquad [\vec{\mathbf{L}}^2, L_\pm] = 0$$

Spin $\frac{1}{2}$: $\vec{\mathbf{S}} = \dfrac{\hbar}{2}\,\vec{\sigma} \qquad |\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \chi_+ = \uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle = \chi_- = \downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_i^2 = 1 \qquad \sigma_x\sigma_y = i\sigma_z = -\sigma_y\sigma_x$$

Clebsch-Gordan: $|jm\rangle = \sum C(jm;\ell m_\ell, s m_s)\,|\ell m_\ell\rangle\,|s m_s\rangle$ know how to use table!

Radial Equation: $\psi(r,\theta,\phi) = Y_{\ell m}(\theta,\phi)R(r) \qquad R(r) = \dfrac{u(r)}{r}$

$$\left[\frac{-\hbar^2}{2m}\left(\partial_r^2 + \frac{2}{r}\partial_r\right) + \frac{\hbar^2\ell(\ell+1)}{2mr^2} + V(r)\right]R = E\,R \qquad \left[\frac{-\hbar^2}{2m}\partial_r^2 + \frac{\hbar^2\ell(\ell+1)}{2mr^2} + V(r)\right]u = E\,u$$

H atom: $H = \dfrac{p^2}{2m} - \dfrac{Ze^2}{4\pi\epsilon_0 r} \qquad E_n = -\dfrac{1}{2}\,mc^2\,\dfrac{(Z\alpha)^2}{n^2} = -\dfrac{1}{2}\,\dfrac{Z^2e^2}{4\pi\epsilon_0 a_0 n^2} \approx -13.6\text{ eV}\ \dfrac{Z^2}{n^2} \qquad n = 1,2,3,\ldots$

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2} \approx .53\text{ Å} \qquad \alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137} \qquad n = n_r + \ell + 1\ \therefore\ 0 \leq \ell \leq n-1 \qquad \rho = \sqrt{\frac{8m|E|}{\hbar^2}}\ r = \frac{2Zr}{na_0}$$

$|n\ell m\rangle = R_{n\ell}(\rho)\,Y_{\ell m}(\theta,\phi)$ where $R_{n\ell} = N_{n\ell}\,\rho^\ell\,L_{n_r}^{2\ell+1}(\rho)\,e^{-\frac{1}{2}\rho} \qquad N_{n\ell} = \dfrac{2}{n^2}\sqrt{\dfrac{(n-\ell-1)!}{(n+\ell)!}}$

Spectroscopic Notation: orbital: s,p,d,f,g term: ${}^{2S+1}L_J$ atomic: 1s,2s,2p,3s,3p,4s,3d nuclear: 1s,1p,1d,2s,1f,2p,1g

2-particle CM Coordinates:

$$\vec{\mathbf{R}} = \frac{m_1}{M}\,\vec{\mathbf{r}}_1 + \frac{m_2}{M}\,\vec{\mathbf{r}}_2 \qquad \vec{\mathbf{r}}_1 = \vec{\mathbf{R}} + \frac{m_2}{M}\,\vec{\mathbf{r}} \qquad M = m_1 + m_2$$

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2 \qquad \vec{\mathbf{r}}_2 = \vec{\mathbf{R}} - \frac{m_1}{M}\,\vec{\mathbf{r}} \qquad \mu = \frac{m_1 m_2}{M}$$

$$\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} = \frac{P^2}{2M} + \frac{p^2}{2\mu}$$

Magnetic: $\vec{\mathbf{p}} \to \vec{\mathbf{p}} - q\vec{\mathbf{A}}$ where q is charge, e.g., for electron: $q = -e \qquad \vec{\mathbf{B}} = \vec{\nabla}\times\vec{\mathbf{A}}$ e.g., uniform $\vec{\mathbf{B}}$ from $\vec{\mathbf{A}} = \dfrac{1}{2}\,\vec{\mathbf{B}}\times\vec{\mathbf{r}}$

Approximation Methods:

WKB: $\int k(x)\,dx = (n-\frac{1}{2})\pi$ (two linear turning points) $\quad \hbar k(x) = p(x) = \sqrt{2m(E-V(x))}$

Rayleigh-Ritz: minimize $E = \dfrac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$

Perturbation Theory: $E_n^1 = \langle n|H'|n\rangle \quad E_n^2 = \sum_{k\neq n} \dfrac{|\langle k|H'|n\rangle|^2}{E_n^0 - E_k^0}$ degenerate: diagonalize matrix $\langle i|H'|j\rangle$

Time Dependent: $c_b(t) \simeq -\dfrac{i}{\hbar}\int_0^t H'_{ba}(t')\,e^{i(E_b-E_a)t'/\hbar}\,dt'$

if $H' = V(\mathbf{r})\cos\omega t$ then: $P_{a\to b} \approx \dfrac{|V_{ab}|^2}{\hbar^2}\,\dfrac{\sin^2((\omega_0-\omega)t/2)}{(\omega_0-\omega)^2}$ E1 Unpolarized Light: $R_{a\to b} = \dfrac{\pi\rho(\omega_0)}{3\epsilon_0\hbar^2}\,|q\langle\psi_b|\vec{\mathbf{r}}|\psi_a\rangle|^2$

Golden Rule: transition rate $= \dfrac{\pi}{2}\,\dfrac{|V_{ab}|^2}{\hbar} \times$ density of states

Scattering: $\dfrac{d\sigma}{d\Omega} = \dfrac{\text{hits/sec in detector}}{J d\Omega} = \dfrac{\text{hits/sec in detector}}{nt \text{ beam current } d\Omega} = \dfrac{b}{\sin\theta}\left|\dfrac{db}{d\theta}\right| = |f|^2$ where: $\psi \approx e^{ikz} + f(\theta)\,\dfrac{e^{ikr}}{r}$

$f(\theta) = \dfrac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin(\delta_\ell)P_\ell(\cos\theta) \quad \sigma = \dfrac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2(\delta_\ell) \quad f_{\text{Born}} = \dfrac{-m}{2\pi\hbar^2}\int e^{i\vec{\mathbf{q}}\cdot\vec{\mathbf{r}}_0}V(\vec{\mathbf{r}}_0)\,d^3\mathbf{r}_0 \quad \vec{\mathbf{q}} = \vec{\mathbf{k}}_i - \vec{\mathbf{k}}_f$

$f_{\text{Born}} = f_{\text{Ruth}}F(q)$ where Form Factor: $F(q) = \int e^{i\vec{\mathbf{q}}\cdot\vec{\mathbf{r}}}\rho(r)\,d^3\mathbf{r}$ using normalized charge density: $\rho \ni \quad 1 = \int \rho(r)\,d^3\mathbf{r}$

Spin-statistics: fermion: $s = \frac{1}{2}, \frac{3}{2}, \ldots$ boson: $s = 0, 1, 2, \ldots$

$$u(x_i) = (N!)^{-1/2}\begin{vmatrix} f(x_1) & f(x_2) & f(x_3) & \cdots \\ g(x_1) & g(x_2) & g(x_3) & \cdots \\ h(x_1) & h(x_2) & h(x_3) & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix}$$