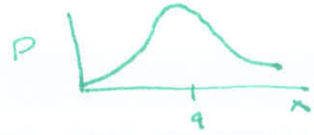
1.3 NB: $\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}}$



$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2}\,dx = \frac{1}{2}\alpha^{-3/2}\sqrt{\pi}$$

$$1 = A\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2}\,dx \underset{x-a=u}{=} A\int_{-\infty}^{\infty} e^{-\lambda u^2}\,du = A\sqrt{\frac{\pi}{\lambda}}$$

$\Rightarrow A = \sqrt{\frac{\lambda}{\pi}}$ check units: note $\lambda = \frac{1}{L^2}$ & $A = \frac{1}{L}$ is expected

$$\langle x\rangle = A\int_{-\infty}^{\infty} x\, e^{-\lambda(x-a)^2}\,dx \underset{\substack{x-a=u\\ x=u+a}}{=} A\int_{-\infty}^{\infty}(u+a)\,e^{-\lambda u^2}\,du = a\,\underbrace{A\int_{-\infty}^{\infty} e^{-\lambda u^2}\,du}_{=1}$$

$u \to 0$

$$= a$$

$$\langle x^2\rangle = A\int_{-\infty}^{\infty} x^2 e^{-\lambda(x-a)^2}\,dx = A\int_{-\infty}^{\infty}(u+a)^2 e^{-\lambda u^2}\,du$$

$$= A\int_{-\infty}^{\infty}\left[u^2 + 2au + a^2\right]e^{-\lambda u^2}\,du = \frac{1}{2\lambda} + a^2$$

$u^2 \to \frac{1}{2\lambda}$, $a^2 \to a^2$

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{1}{2\lambda} + a^2 - a^2 = \frac{1}{2\lambda} \Rightarrow \sigma_x = \frac{1}{\sqrt{2\lambda}}$$

1.4 Recall $\rho = \Psi^*\Psi = \begin{cases} A^2\left(\frac{x}{a}\right)^2 \\ A^2\left(\frac{b-x}{b-a}\right)^2 \end{cases}$

$$1 = \int_{-\infty}^{\infty}\rho\,dx = A^2\left[\int_0^a \frac{x^2}{a^2}\,dx + \int_a^b \left(\frac{b-x}{b-a}\right)^2 dx\right]$$

$$= A^2\left[\frac{1}{3}a + \frac{1}{3}(b-a)\right] = A^2\frac{b}{3} \Rightarrow A = \sqrt{\frac{3}{b}}$$

check units: expect $[\Psi] = \frac{1}{\sqrt{L}} = [A]$ correct as $[b] = L$

Prob $x \in [0,a] = \int_0^a \rho\,dx = A^2\int_0^a \frac{x^2}{a^2}\,dx = A^2\frac{a}{3} = \frac{a}{b}$

check $b = a \Rightarrow 1$ $\quad b = 2a = 1/2$ makes sense

$$\langle x\rangle = \int_{-\infty}^{\infty} x\rho\,dx = A^2\left[\int_0^a \frac{x^3}{a^2}\,dx + \int_a^b \frac{x(b-x)^2}{(b-a)^2}\,dx\right]$$

$$= \frac{3}{b}\left[\frac{a^2}{4} + \frac{1}{(b-a)^2}\int_a^b b(b-x)^2 - (b-x)^3\right]$$

$\to \frac{1}{3}b(b-a)^3 - \frac{1}{4}(b-a)^4$

$\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

$-\frac{1}{3} + \frac{1}{2} = \frac{1}{6}$

$$= \frac{3}{b}\left[\frac{a^2}{4} + \frac{1}{3}b(b-a) - \frac{1}{4}(b-a)^2\right] = \frac{3}{b}\left[\frac{1}{12}b^2 + \frac{1}{6}ba\right]$$

$$= \left[\frac{b+2a}{4}\right]$$

check: $b = 2a \Rightarrow \langle x\rangle = a$ which makes sense

2-4 $\quad \psi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$

$$\langle x\rangle = \int_0^a \psi^* x \psi = \frac{2}{a}\int_0^a x \sin^2\Big(\underbrace{\frac{n\pi}{a}x}_{u}\Big)\,dx = \frac{2a}{(n\pi)^2}\int_0^{n\pi} u \sin^2 u\, du$$

$$= \frac{2a}{(n\pi)^2}\left[\frac{u^2}{4} - \frac{u \sin 2u}{4} - \frac{\cos 2u}{8}\right]_0^{n\pi} = \frac{2a}{(n\pi)^2}\,\frac{(n\pi)^2}{4} = \frac{a}{2} \checkmark$$

$$\frac{u\sin 2u}{4} \to 0 \qquad \frac{\cos 2u}{8} \to 0$$

$$\langle x^2\rangle = \int_0^a \psi^* x^2 \psi\, dx = \frac{2}{a}\int_0^a x^2 \sin^2\left(\frac{n\pi}{a}x\right)dx = \frac{2a^2}{(n\pi)^3}\int_0^{n\pi} u^2 \sin^2 u\, du$$

$$= \frac{2a^2}{(n\pi)^3}\left[\frac{u^3}{6} - \left(\frac{u^2}{4} - \frac{1}{8}\right)\sin 2u - \frac{x\cos 2u}{4}\right]_0^{n\pi}$$

$$\sin 2u \to 0 \qquad \cos 2\pi n = 1$$

$$= \frac{2a^2}{(n\pi)^3}\left[\frac{(n\pi)^3}{6} - \frac{(n\pi)}{4}\right] = \frac{a^2}{3}\left[1 - \frac{3}{2}\frac{1}{(n\pi)^2}\right]$$

$$\langle p\rangle = \int_0^a \psi^* \frac{\hbar}{i}\partial_x \psi\, dx \underset{\text{since } \psi \text{ is Real here}}{=} \frac{\hbar}{i}\int_0^a \psi\,\partial_x\psi\, dx = \frac{\hbar}{i}\int_0^a \partial_x\left(\frac{\psi^2}{2}\right)dx$$

Note: if ψ is real $\langle p\rangle$ must be zero!

$$= \frac{\hbar}{i}\,\psi^2\Big|_0^a = 0$$

$$\langle p^2\rangle = -\hbar^2\int_0^a \psi\psi''\,dx = -\hbar^2\frac{2}{a}\int_0^a \sin(kx)\left[-k^2\sin(kx)\right]dx$$

$$= +\hbar^2\frac{2}{a}k^2\underbrace{\int_0^a \sin^2(kx)\,dx}_{\frac{1}{2}a} = \hbar^2k^2 = \hbar^2\left(\frac{n\pi}{a}\right)^2$$

$$\sigma_p^2 = \langle p^2\rangle$$

$$\sigma_x^2 = \frac{a^2}{3}\left[1 - \frac{3}{2}\frac{1}{(n\pi)^2}\right] - \left(\frac{a}{2}\right)^2 = \frac{1}{12}a^2 - \frac{a^2}{2(n\pi)^2}$$

$$= \frac{a^2}{12}\left[1 - \frac{6}{(n\pi)^2}\right]$$

$$\sigma_x^2\sigma_p^2 = \frac{a^2}{12}\left[1 - \frac{6}{(n\pi)^2}\right]\hbar^2\left(\frac{n\pi}{a}\right)^2 = \frac{\hbar^2}{12}\left[(n\pi)^2 - 6\right]$$

clearly smallest value is at $n=1$

for $n=1$ $\quad \sigma_x\sigma_p = .6\,\hbar$

2.5

$$1 = \int \Psi^* \Psi \, dx = A^2 \int (\psi_1+\psi_2)^* (\psi_1+\psi_2)\, dx = A^2 \int \underbrace{\psi_1^*\psi_1}_{1} + \underbrace{\psi_1^*\psi_2}_{0} + \underbrace{\psi_2^*\psi_1}_{0} + \underbrace{\psi_2^*\psi_2}_{1}$$

$$= A^2 \cdot 2 \Rightarrow A = \frac{1}{\sqrt{2}}$$

$$\left.\begin{array}{l} E_1 = \frac{\hbar^2 \pi^2}{2ma^2} \\ E_2 = 4 \cdot E_1 \end{array}\right\} E_2 - E_1 = 3E_1 = 3\hbar\omega$$

$$\Psi = \frac{1}{\sqrt{2}}\left(\psi_1 e^{-i\frac{E_1 t}{\hbar}} + \psi_2 e^{-i\frac{E_2 t}{\hbar}}\right)$$ Note ψ_n are real here.

$$\Psi^*\Psi = \frac{1}{2}\left(\psi_1^2 + \psi_2^2 + \psi_1\psi_2 \underbrace{\left(e^{i\frac{(E_2-E_1)t}{\hbar}} + e^{-i\frac{(E_2-E_1)t}{\hbar}}\right)}_{2\cos\left[\frac{(E_2-E_1)t}{\hbar}\right] = 2\cos 3\omega t}\right)$$

$$= \frac{1}{2}\left(\psi_1^2 + \psi_2^2 + 2\psi_1\psi_2 \cos(3\omega t)\right)$$

$$\langle x \rangle = \int_0^a x \, \frac{1}{2}\left(\underbrace{\psi_1^2}_{\frac{a}{2}} + \underbrace{\psi_2^2}_{\frac{a}{2}} + 2\psi_1\psi_2 \cos(3\omega t)\right) dx$$

as $\langle x \rangle = \frac{a}{2}$ for stationary states

$$= \underbrace{\frac{2}{a}\int_0^a x \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right) dx}\ \cos(3\omega t) + \frac{a}{2}$$

$$\hookrightarrow \frac{a^2}{\pi^2}\int_0^\pi u \sin u \sin(2u)\, du = \frac{a^2}{\pi^2}\int_0^\pi u \; 2\sin^2 u \cos u \, du$$

$$= \frac{a^2}{\pi^2}\int_0^\pi u \, \partial_u (\sin^3 u) \frac{2}{3} du \underset{\text{parts}}{=} -\frac{a^2}{\pi^2}\int_0^\pi \sin^3 u \, \frac{2}{3} du$$

$$= -\frac{a^2}{\pi^2}\frac{2}{3}\left[\underbrace{\frac{\cos^3 u}{3}}_{-\frac{2}{3}} - \underbrace{\cos u}_{2}\right]_0^\pi = -\frac{a^2}{\pi^2}\frac{8}{9}$$

$$= \frac{a}{2} - \frac{16a}{9\pi^2}\cos(3\omega t)$$

$\frac{16a}{9\pi^2}$ ← amplitude; 3ω ← angular freq

easy: by Eq 1.33 $\langle p \rangle = m\frac{d\langle x \rangle}{dt} = m 3\omega \frac{16a}{9\pi^2}\sin(3\omega t)$

Hard: $$\langle p \rangle = A^2 \int \left[\underbrace{\psi_1^* p \psi_1}_{0} + \psi_1^* p \psi_2 + \psi_2^* p \psi_1 + \underbrace{\psi_2^* p \psi_2}_{0}\right] dx$$

$$= A^2 \left\{ \int \sin(k_1 x)\partial_x \sin(k_2 x) e^{i\frac{(E_1-E_2)t}{\hbar}} + \int \sin(k_2 x)\partial_x \sin(k_1 x)\, e^{i\frac{(E_1-E_2)t}{\hbar}} \right\} \frac{2}{a}\frac{\hbar}{i}$$

$\hookrightarrow$ parts

$$= A^2 \left\{ \int \sin(k_2 x)\partial_x \sin(k_1 x)\, dx \underbrace{\left[e^{i\frac{E_2-E_1}{\hbar}t} - e^{-i\frac{E_2-E_1}{\hbar}t}\right]}_{2i\sin(3\omega t)} \right\} \frac{2}{a}\frac{\hbar}{i}$$

2-5 cont $\langle p\rangle = A^2 \frac{2}{a}\frac{\hbar}{i} 2i \sin(3\omega t) \int_0^a \sin\left(\frac{2\pi}{a}x\right) \frac{\pi}{a}\cos\left(\frac{\pi}{a}x\right) dx$

$u = \frac{\pi}{a}x \qquad \sin(2\theta) = 2\sin\theta\cos\theta$

$= A^2 \frac{4}{a}\hbar \sin(3\omega t)\frac{4}{3}$

$\int_0^\pi 2\sin(u)\cos(u)\cos(u)\,du$

$= \frac{8}{3a}\hbar \sin(3\omega t)$

$= 2\int_0^\pi \cos^2 u \sin u\,du$ $\qquad X = \cos u$

$\qquad dX = -\sin u\,du$

$= 2\int_1^{-1} X^2(-dX) = \frac{4}{3}$

Note: $m\omega a = m\frac{\hbar\pi^2}{2ma^2}a$

$= \frac{\hbar}{a}\frac{\pi^2}{2}$

so same result as "easy"

$c_1 = \frac{1}{\sqrt{2}} \quad c_2 = \frac{1}{\sqrt{2}}$ other $c = 0 \Rightarrow \langle H\rangle = c_1^2E_1 + c_2^2E_2$

equally likely: Prob $= \frac{1}{2}$ for E_1 & E_2; zero for all other E

2-7 

$\frac{1}{2} = \int_0^{a/2}\psi^2 dx = A^2\int_0^{a/2}x^2 dx = A^2\frac{(a/2)^3}{3}$

$\frac{12}{a^3} = A^2$

$\Psi(x,t) = \sum c_n \psi_n e^{-iE_nt/\hbar}$

$\int \psi_n^* \psi_{start}\,dx$

Note: since the starting wave function is even (reflects thru $a/2$) it must be composed only from even functions, so $c_n = 0$ for $n =$ even. Furthermore for these "even" functions: $\int_0^a = 2\int_0^{a/2}$

$c_n = \sqrt{\frac{2}{a}}\,2\int_0^{a/2}\sin\left(\frac{n\pi}{a}x\right)Ax\,dx$

$= \sqrt{\frac{2}{a}}\,2A\left(\frac{a}{n\pi}\right)^2\int_0^{n\pi/2}\sin u\,u\,du$

$\hookrightarrow \left[\sin u - u\cos u\right]_0^{n\pi/2} = 1$ or -1 ← alternates

$= \sqrt{\frac{2}{a}}\,2\sqrt{\frac{12}{a^3}}\left(\frac{a}{n\pi}\right)^2(\pm 1)$

$= \frac{4\sqrt{6}}{\pi^2 n^2}(\pm 1) \qquad P_1 = c_1^2 = \frac{16\cdot 6}{\pi^4}$

Dwight 48.12 $\quad \pi^2/8$

$\langle E\rangle = \sum_{n\ odd}\frac{16\cdot 6}{\pi^4 n^4}\frac{\hbar^2\pi^2n^2}{2ma^2} = \frac{48\hbar^2}{\pi^2 ma^2}\sum_{odd}\frac{1}{n^2} = \frac{6\hbar^2}{ma^2}$

or Sum[1/n^2, {n,1,Infinity,2}]

alt: $\langle H\rangle = \int\psi^*\frac{p^2}{2m}\psi\,dx \overset{\text{trick!}}{=} \frac{1}{2m}\int(p\psi)^*(p\psi)\,dx$

$= \frac{1}{2m}2\int_0^{a/2}A^2dx\left(\frac{\hbar}{i}\right)^*\left(\frac{\hbar}{i}\right) = \frac{\hbar^2}{m}A^2\frac{a}{2} = \frac{6\hbar^2}{ma^2}$ ✓

For H_{10} $E' = 21$

$$a_2 = -\frac{21-1}{2} a_0 = -10\, a_0$$ ✓

$$a_4 = -\frac{21-5}{4 \cdot 3} a_2 = \frac{-4}{3} a_2$$ ✓

$$a_6 = -\frac{21-9}{6 \cdot 5} a_4 = -\frac{2}{5} a_4$$ ✓

$$a_8 = -\frac{21-13}{8 \cdot 7} a_6 = \frac{-1}{7} a_6$$ ✓

$$a_{10} = -\frac{21-17}{10 \cdot 9} a_8 = \frac{-2}{45} a_8$$ ✓

see class3.m

2-12 recall: $x = \sqrt{\frac{\hbar}{m\omega}} \frac{1}{\sqrt{2}}(a_+ + a_-)$ $\qquad a_+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$p = i\sqrt{\hbar m\omega}\,\frac{1}{\sqrt{2}}(a_+ - a_-)$ $\qquad a_-|n\rangle = \sqrt{n}\,|n-1\rangle$

$\langle m|n\rangle = \delta_{mn}$

$\langle x\rangle = 0$ expected as symmetric potential
$\langle p\rangle = 0$ expected as ψ is real
} confirm as $\langle n|\underbrace{a_\pm|n\rangle}_{\sim|n\pm1\rangle} = 0$

$\langle x^2\rangle = \frac{\hbar}{m\omega}\frac{1}{2}\langle n|(a_+ + a_-)^2|n\rangle$

$\rightarrow \underbrace{a_+^2}_{0} + a_+a_- + a_-a_+ + \underbrace{a_-^2}_{0}$

$= \frac{\hbar}{m\omega}\frac{1}{2}\langle n|\,a_+a_- + a_-a_+\,|n\rangle$

$a_+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$a_-|n\rangle = \sqrt{n}\,|n-1\rangle$

$a_-a_+|n\rangle = \sqrt{n+1}\sqrt{n+1}\,|n\rangle = (n+1)|n\rangle$

$a_+a_-|n\rangle = \sqrt{n}\sqrt{n}\,|n\rangle = n|n\rangle$

$= \frac{\hbar}{m\omega}\frac{1}{2}(n + n + 1) = \frac{\hbar}{m\omega}(n + 1/2)$

$\langle p^2\rangle = -(\hbar m\omega)\frac{1}{2}\langle n|(a_+ - a_-)^2|n\rangle$

$\rightarrow \underbrace{a_+a_+}_{0} - a_+a_- - a_-a_+ + \underbrace{a_-^2}_{0}$

$= +(\hbar m\omega)\frac{1}{2}\langle n|\underbrace{a_+a_- + a_-a_+\,|n\rangle}_{\text{as before } = 2n+1}$

$= (\hbar m\omega)(n + 1/2)$

$\langle T\rangle = \frac{1}{2m}\langle p^2\rangle = \frac{\hbar\omega}{2}(n + 1/2)$

FYI: $\langle V\rangle = \frac{1}{2}k\langle x^2\rangle = \frac{1}{2}m\omega^2\langle x^2\rangle = \frac{\hbar\omega}{2}(n + 1/2)$ also

$\Delta x\,\Delta p = \sqrt{\frac{\hbar}{m\omega}}\sqrt{\hbar m\omega}\,(n + 1/2) = \hbar(n + 1/2)$

2-13 (very similar to 2-5) $\quad$ ⑥

$$1 = \langle \psi | \psi \rangle = |A|^2 \langle 3\psi_0 + 4\psi_1 | 3\psi_0 + 4\psi_1 \rangle$$

$$= A^2 \left\{ 9 \underbrace{\langle \psi_0 | \psi_0 \rangle}_{1} + 12 \left[\underbrace{\langle \psi_0 | \psi_1 \rangle}_{0} + \underbrace{\langle \psi_1 | \psi_0 \rangle}_{0} \right] + 16 \underbrace{\langle \psi_1 | \psi_1 \rangle}_{1} \right\}$$

$$= A^2 \cdot 25 \Rightarrow A = 1/5$$

$$\psi = c_0 \psi_0 e^{-\frac{iE_0 t}{\hbar}} + c_1 \psi_1 e^{-\frac{iE_1 t}{\hbar}}$$ where $c_0 = A \cdot 3 = 3/5$, $c_1 = A \cdot 4 = 4/5$

$$|\psi|^2 = |c_0|^2 |\psi_0|^2 + |c_1|^2 |\psi_1|^2 + c_1^* c_0 \psi_1^* \psi_0 e^{i\omega t} + \psi_0^* \psi_1 e^{-i\omega t} c_0^* c_1$$

$E_i = \hbar\omega(i + 1/2)$

$E_1 - E_0 = \hbar\omega$

$$\langle x \rangle = |c_0|^2 \underbrace{\langle 0 | x | 0 \rangle}_{0} + |c_1|^2 \underbrace{\langle 1 | x | 1 \rangle}_{0}$$

$$+ c_1^* c_0 \underbrace{\langle 1 | x | 0 \rangle}_{\sqrt{\frac{\hbar}{m\omega}} \frac{1}{\sqrt{2}}} e^{i\omega t} + c_0^* c_1 \underbrace{\langle 0 | x | 1 \rangle}_{\sqrt{\frac{\hbar}{m\omega}} \frac{1}{\sqrt{2}}} e^{-i\omega t}$$

using $a_+|0\rangle = |1\rangle$ $\qquad$ using $a_-|1\rangle = |0\rangle$

$$= \frac{12}{25} \sqrt{\frac{\hbar}{m\omega}} \sqrt{2} \left(\underbrace{\frac{e^{i\omega t} + e^{-i\omega t}}{2}}_{\cos(\omega t)} \right)$$

$$\langle p \rangle = |c_0|^2 \langle 0 | p | 0 \rangle + |c_1|^2 \langle 1 | p | 1 \rangle$$

$$+ c_1^* c_0 \underbrace{\langle 1 | p | 0 \rangle}_{i\sqrt{\hbar m\omega} \frac{1}{\sqrt{2}}} e^{i\omega t} + c_0^* c_1 \underbrace{\langle 0 | p | 1 \rangle}_{-i\sqrt{\hbar m\omega} \frac{1}{\sqrt{2}}} e^{-i\omega t}$$

using $a_+|0\rangle = |1\rangle$ $\qquad$ using $|a_-|1\rangle = |0\rangle$

$$= \frac{12}{25} \sqrt{\hbar m\omega} \sqrt{2} \left(\underbrace{i\frac{e^{i\omega t} - i e^{-i\omega t}}{2}}_{-\sin(\omega t)} \right)$$

indeed $\frac{d\langle x \rangle}{dt} = \frac{\langle p \rangle}{m}$ $\qquad$ $\langle -\partial_x V \rangle = \langle -kx \rangle = -m\omega^2 \langle x \rangle = \frac{d\langle p \rangle}{dt}$ ✓

E_0 with Prob $= c_0^2 = \frac{9}{25}$ $\qquad$ E_1 with prob $= c_1^2 = \frac{16}{25}$

2-19 Note time dependent part: $e^{-i\omega t}$ will cancel in $\psi^*\psi$

$$J = \frac{\hbar}{2mi}\psi^* \overleftrightarrow{\partial}\psi = \frac{\hbar}{2mi}\left[ik|A|^2 - -ik|A|^2\right] = \frac{\hbar k |A|^2}{m} \quad \text{"}\frac{p}{m}\text{"} = \text{"}v\text{"}$$ ✓

2-21 $$1 = \int_{-\infty}^{\infty} A^2 e^{-2q|x|}\,dx = 2\int_0^\infty A^2 e^{-2qx}\,dx = 2A^2\left[\frac{e^{-2qx}}{-2q}\right]_0^\infty$$

$$= \frac{A^2}{q} \Rightarrow A = \sqrt{q} \quad \text{check units: } \frac{1}{\sqrt{L}}$$

$$\phi(k) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-ikx} A e^{-q|x|}\,dx = \sqrt{\frac{q}{2\pi}}\left\{\int_{-\infty}^{0} e^{(q-ik)x}\,dx + \int_0^\infty e^{(-q-ik)x}\,dx\right\}$$

$$= \sqrt{\frac{q}{2\pi}}\left\{\frac{1}{q-ik} + \frac{1}{q+ik}\right\} = \sqrt{\frac{q}{2\pi}}\left\{\frac{2q}{q^2+k^2}\right\}$$

$$\psi = \int_{-\infty}^{\infty} \phi(k)\frac{1}{\sqrt{2\pi}} e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)}\,dk$$

$$\Delta x\,\Delta k \sim \frac{1}{q}\cdot q \sim 1$$

2-24. clearly $\delta(cx)$ is delta function like: zero except at $x=0$

but is $\int \delta(cx)\,dx = 1$? (No)

$$\int_{-\infty}^{\infty} \delta(cx)\,dx = \frac{1}{c}\int_{-\infty}^{\infty} \underset{u}{\delta(cx)}\, \underset{du}{c\,dx} = \frac{1}{c}\left\{\begin{array}{l} c>0: \int_{-\infty}^{\infty}\delta\, du = 1 \\ c<0: \int_{\infty}^{-\infty}\delta(u)\,du = -1\end{array}\right\} = \frac{1}{|c|}$$

So $\delta(cx)$ acts just like $\frac{1}{|c|}\delta(x)$

check units: $[\delta(cx)] = \frac{1}{cx}$ $\quad [\delta(x)] = \frac{1}{x}$ ✓

2-34 a) $e^{ikx} + Re^{-ikx}$ | Te^{-kx}

$E = \frac{\hbar^2 k^2}{2m}$ | $E = -\frac{\hbar^2 k^2}{2m} + V_0$

ψ cont: $1 + R = T$

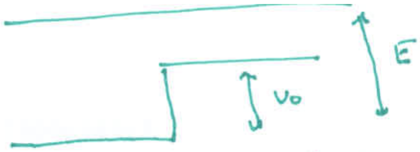
ψ' cont: $ik(1-R) = -kT$

$$\left.\right\} \quad 1 + R = \frac{ik}{k}(R-1) \Rightarrow \frac{1 + \frac{ik}{k}}{-1 + \frac{ik}{k}} = R$$

$$|R|^2 = 1$$ ✓

Faster: since on rhs ψ = real, Flux = 0 hence incoming / outgoing flux must be same

2-34



$1 + Re^{-ikx}$ $\qquad Te^{ilx}$

$E = \dfrac{\hbar^2 k^2}{2m}$ $\qquad \dfrac{\hbar^2 l^2}{2m} + V_0 = E$

$\dfrac{l}{k} = \sqrt{\dfrac{E - V_0}{E}}$

$\dfrac{l^2}{k^2} = \dfrac{E - V_0}{E} \longleftarrow$ $\qquad \dfrac{\;\dfrac{\hbar^2 l^2}{2m} = E - V_0\;}{\dfrac{\hbar^2 k^2}{2m} = E}$

$$\left.\begin{array}{l} \text{Flux: } \Rightarrow 1^2 \dfrac{\hbar k}{m} \qquad \Rightarrow |T|^2 \dfrac{\hbar l}{m} \\ \qquad \Leftarrow |R|^2 \dfrac{\hbar k}{m} \end{array}\right\} \text{transmission} = \frac{|T|^2 \frac{\hbar l}{m}}{|1|^2 \frac{\hbar k}{m}} = |T|^2 \frac{l}{k}$$

$$\left.\begin{array}{ll} \psi \text{ cont: } & 1 + R = T \\ \psi' \text{ cont: } & ik(1 - R) = ilT \end{array}\right\} \quad \begin{array}{l} 1 + R = T \\ 1 - R = \frac{l}{k} T \\ \hline 2 = \left(1 + \frac{l}{k}\right) T \end{array}$$

$$\frac{2}{(1 + l/k)} = T$$

or $1 + R = \frac{k}{l}(1 - R)$

$\left(1 - \frac{k}{l}\right) = -R\left(1 + \frac{k}{l}\right)$

$$\frac{\frac{k}{l} - 1}{\frac{k}{l} + 1} = R$$

$$\parallel$$

$$\frac{1 - l/k}{1 + l/k}$$

$$\left.\begin{array}{l} \%\text{ transmiss.} = \dfrac{4}{(1 + l/k)^2} \dfrac{l}{k} \\ \%\text{ reflect} = |R|^2 = \dfrac{(1 - l/k)^2}{(1 + l/k)^2} \end{array}\right\} \quad \frac{\left(1 - \frac{l}{k}\right)^2 + 4\frac{l}{k}}{(1 + l/k)^2} = \frac{(1 + l/k)^2}{(1 + l/k)^2} = 1$$

A-8 $\quad A + B = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \\ 3i & 3-2i & 4 \end{pmatrix}$

$AB = \begin{pmatrix} -1 & 1 & i \\ 2 & 0 & 3 \\ 2i & -2i & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & -i \\ 0 & 1 & 0 \\ i & 3 & 2 \end{pmatrix} = \begin{pmatrix} -3 & 1+3i & 3i \\ 4+3i & 9 & 6-2i \\ 6i & 6-2i & 6 \end{pmatrix}$

$A^T = \tilde{A} = \begin{pmatrix} -1 & 2 & 2i \\ 1 & 0 & -2i \\ i & 3 & 2 \end{pmatrix}$

$BA = \begin{pmatrix} 2 & 0 & -i \\ 0 & 1 & 0 \\ i & 3 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 & i \\ 2 & 0 & 3 \\ 2i & -2i & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 3 \\ 6+3i & -3i & 12 \end{pmatrix}$

$A^* = \begin{pmatrix} -1 & 1 & -i \\ 2 & 0 & 3 \\ -2i & +2i & 2 \end{pmatrix}$

$[A, B] = AB - BA \qquad \begin{pmatrix} -3 & 1+3i & 3i \\ 2+3i & 9 & 3-2i \\ -6+3i & 6+i & -6 \end{pmatrix}$

$A^\dagger = \begin{pmatrix} -1 & 2 & -2i \\ 1 & 0 & +2i \\ -i & 3 & 2 \end{pmatrix}$

$B^{-1} = \dfrac{1}{\det B} \begin{pmatrix} 2 & -3i & i \\ 0 & 3 & 0 \\ -i & -6 & 2 \end{pmatrix} \qquad -\begin{vmatrix} 0 & -i \\ 3 & 2 \end{vmatrix}$

$\operatorname{Tr} B = 5$

Cramer's Rule: sub determinants from transpose with signs

$\det B = 4 + 0 + 0 - 1 = 3$ $\qquad -\begin{vmatrix} 0 & 0 \\ i & 2 \end{vmatrix}$

note: inverse 2x2: $\dfrac{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}{ad - bc}$

A-9

$$Aa = \begin{pmatrix} -1 & 1 & i \\ 2 & 0 & 3 \\ 2i & -2i & 2 \end{pmatrix} \begin{pmatrix} i \\ 2i \\ 2 \end{pmatrix} = \begin{pmatrix} 3i \\ 6+2i \\ 6 \end{pmatrix}$$

$$a^{+}b = (-i, -2i, 2) \begin{pmatrix} 2 \\ 1-i \\ 0 \end{pmatrix} = -2i \; -2i \; -2 \; = -2-4i$$

$$a^{T}Bb = (i \; 2i \; 2) \begin{pmatrix} 2 & 0 & -i \\ 0 & 1 & 0 \\ i & 3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1-i \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 \\ 1-i \\ 3+i \end{pmatrix} = 4i + 2i + 2 = 8+4i$$

$$6-2i$$

$$ab^{+} = \begin{bmatrix} i \\ 2i \\ 2 \end{bmatrix} [2, 1+i, 0] = \begin{pmatrix} 2i & i-1 & 0 \\ 4i & 2i-2 & 0 \\ 4 & 2+2i & 0 \end{pmatrix}$$

A-26 $\det T = 8 -1 -1 -2 -2 -2 = 0$ $(\Rightarrow \exists$ a zero eigenvalue$)$

$\mathrm{Tr}\, T = 6$

$$\det \begin{bmatrix} 2-\lambda & i & 1 \\ -i & 2-\lambda & i \\ 1 & -i & 2-\lambda \end{bmatrix} = (2-\lambda)^3 - 1 - 1 - (2-\lambda) - (2-\lambda) - (2-\lambda)$$

$$= 8 - 12\lambda + 6\lambda^2 - \lambda^3 - 8 + 3\lambda$$

$$= 9\lambda + 6\lambda^2 - \lambda^3 = -\lambda(\lambda-3)^2$$

$\uparrow 0$ $\quad$ $\nwarrow$ double root: 3

seek vector for $\lambda = 0$

$$\begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \qquad \lambda = 0 \rightarrow \begin{pmatrix} -1 \\ -i \\ 1 \end{pmatrix} = \vec{w_0}$$

this row is $row_1 - i\, row_2$

add $-2i \cdot row_2$ to row_1

$$\begin{pmatrix} 0 & -3i & +3 \\ -i & 2 & i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \rightarrow \text{so } v_2 = -i v_3$$

say $v_3 = 1 \Rightarrow v_2 = -i$

$row_2 \Rightarrow v_1 = -1$

Note: the other eigenvectors must be orthogonal to this (in fact since the 2d space $\perp$ to $\vec{w_0}$ ~~must~~ any vector $\perp$ to $\vec{w_0}$ must be — I'll use $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ which checks

For my 3RD eigenvector, I select something $\perp$ to both

$$\vec{w_0}^{*} \times \vec{w_1} \sim \begin{pmatrix} -1 \\ 2i \\ 1 \end{pmatrix}$$

$$\sim 1 + x_0\partial_x + \tfrac{1}{2!}x_0^2\partial_x^2 + \tfrac{1}{3!}x_0^3\partial_x^3$$

3-39 $$e^{i x_0 \frac{p}{\hbar}} f = e^{x_0 \partial_x} f(x) = f(x) + x_0 f'(x) + \tfrac{1}{2} x_0^2 f''(x) + \ldots$$
$$= f(x+x_0) \quad \text{by Taylor Expansion}$$

$$e^{-i t_0 H/\hbar}\,\psi = e^{+t_0 \partial_t}\,\psi \quad \checkmark$$

Online Commutators

$$[p,x^2]f = px^2 f - x^2 p f = \frac{\hbar}{i}2x f + x^2 p f - x^2 p f = \frac{\hbar}{i}2x$$
$$[p,V]f = pVf - Vpf = \frac{\hbar}{i}V' f + Vpf - Vpf = \frac{\hbar}{i}V' f$$

$$\left.\begin{aligned}[p^2,x] &= p[p,x] + [p,x]p = 2\frac{\hbar}{i}p \\ [p,x] &= \frac{\hbar}{i}p^0\end{aligned}\right\}\ \text{cases } n = 1 \;\&\; 2$$

To show case $n \Rightarrow$ case $n+1$

$$[p^n,x] = p[p^{n-1},x] + [p,x]p^{n-1} = p\,(n-1)\frac{\hbar}{i}p^{n-2} + \frac{\hbar}{i}p^{n-1}$$
$$= n\frac{\hbar}{i}p^{n-1} \quad \checkmark$$

3-31 $$\Big[\underbrace{\tfrac{p^2}{2m}+V}_{H}, xp\Big] = \frac{1}{2m}\left\{[p^2,xp]\right\} + [V,xp]$$

$$[p^2,xp] \to \underbrace{[p^2,x]}_{2\frac{\hbar}{i}p}p + x\underbrace{[p^2,p]}_{\to 0}$$

$$[V,xp] \to \underbrace{[V,x]}_{\to 0}p + x\underbrace{[V,p]}_{\to -\frac{\hbar}{i}V'}$$

$$= \frac{\hbar}{mi}xp^2 - \frac{\hbar}{i}xV'$$

$$\frac{d}{dt}\langle xp\rangle = \Big\langle \underbrace{\frac{\partial xp}{\partial t}}_{\to 0}\Big\rangle + \Big\langle \underbrace{\tfrac{1}{m}p^2}_{\to 2T} - xV'\Big\rangle = \langle 2T\rangle - \langle xV'\rangle$$

stationary state $\Rightarrow \psi = e^{-iEt/\hbar}$ so $\psi^*\psi \sim e^0$ so $\frac{d}{dt}\langle\ \rangle = 0$

For SHO: $V = \frac{1}{2}kx^2 \qquad xV' = kx^2 = 2V$

3-37 $\quad H = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix}$ eigen values / vectors

$$\begin{array}{ccc} c & a+b & a-b \\ \begin{pmatrix}0\\1\\0\end{pmatrix} & \begin{pmatrix}1\\0\\1\end{pmatrix} & \begin{pmatrix}1\\0\\-1\end{pmatrix} \\ \psi_0 & \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} & \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix} \\ & \psi_+ & \psi_- \end{array}$$

(a) $\quad |S(t)\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} e^{-ict/\hbar}$

(b) $\quad \begin{pmatrix}0\\0\\1\end{pmatrix} = \alpha_0\psi_0 + \alpha_+\psi_+ + \alpha_-\psi_-$

$\langle\psi_0|\begin{smallmatrix}0\\0\\1\end{smallmatrix}\rangle = 0 \qquad \langle\psi_+|\begin{smallmatrix}0\\0\\1\end{smallmatrix}\rangle = +\frac{1}{\sqrt{2}} \qquad \langle\psi_-|\begin{smallmatrix}0\\0\\1\end{smallmatrix}\rangle = -\frac{1}{\sqrt{2}}$

future: $\quad \frac{1}{\sqrt{2}}\psi_+ e^{-i(a+b)t/\hbar} + \left(-\frac{1}{\sqrt{2}}\right)\psi_- e^{-i(a-b)t/\hbar}$

3-38

H: eigen values / vectors

$$\begin{array}{ccc} \hbar\omega & 2\hbar\omega & 2\hbar\omega \\ \begin{pmatrix}1\\0\\0\end{pmatrix} & \begin{pmatrix}0\\1\\0\end{pmatrix} & \begin{pmatrix}0\\0\\1\end{pmatrix} \end{array}$$

A: eigen values / vectors

$$\begin{array}{ccc} \lambda & -\lambda & 2\lambda \\ \begin{pmatrix}1\\1\\0\end{pmatrix} & \begin{pmatrix}1\\-1\\0\end{pmatrix} & \begin{pmatrix}0\\0\\1\end{pmatrix} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \end{array}$$

B: eigen values / vectors

$$\begin{array}{ccc} 2\mu & \mu & -\mu \\ \begin{pmatrix}1\\0\\0\end{pmatrix} & \begin{pmatrix}0\\1\\1\end{pmatrix} & \begin{pmatrix}0\\1\\-1\end{pmatrix} \\ & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array}$$

$\langle H\rangle = (c_1^* \; c_2^* \; c_3^*) \begin{pmatrix} 1 & & \\ & 2 & \\ & & 2 \end{pmatrix} \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} \hbar\omega = \hbar\omega\left[1|c_1|^2 + 2|c_2|^2 + 2|c_2|^2\right]$

$\langle A\rangle \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \lambda = \lambda\left[c_1^* c_2 + c_2^* c_1 + 2|c_3|^2\right]$

$\langle B\rangle \qquad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \mu = \mu\left[2|c_1|^2 + c_2^* c_3 + c_3^* c_2\right]$

$|c_i|^2$ is prob of measuring E_i $\qquad |S\rangle = c_1\begin{pmatrix}1\\0\\0\end{pmatrix}e^{-i\omega t} + c_2\begin{pmatrix}0\\1\\0\end{pmatrix}e^{-2i\omega t} + c_3\begin{pmatrix}0\\0\\1\end{pmatrix}e^{-2i\omega t}$

To find, say the prob a measurement of B yields $-\mu$, dot $|S\rangle$ with

$$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix} \cdot \left\{ c_1\begin{pmatrix}1\\0\\0\end{pmatrix}e^{-i\omega t} + c_2\begin{pmatrix}0\\1\\0\end{pmatrix}e^{-2i\omega t} + c_3\begin{pmatrix}0\\0\\1\end{pmatrix}e^{-2i\omega t} \right\}$$

$$= \frac{1}{\sqrt{2}}(c_2 - c_3)e^{-2i\omega t}$$

$$\text{Prob} = |\quad|^2 = \frac{1}{2}|c_2 - c_3|^2$$

old exam #2 & #6

$2+2i \quad 1 \quad 4$

(2) $|N|^2 \left(\langle c_1 u_1 + c_2 u_2 + c_3 u_3 | c_1 u_1 + c_2 u_2 + c_3 u_3 \rangle \right)^2$

$= |N|^2 \left(|c_1|^2 + |c_2|^2 + |c_3|^2 \right) = |N|^2 (4+4+1+16) = 25|N|^2 = 1$

$\Rightarrow N = \frac{1}{5}$

$a_- - a_+ = \sqrt{\frac{m\omega}{2\hbar}} \frac{2i}{m\omega} p \qquad p = i(a_+ - a_-)\sqrt{\frac{\hbar m\omega}{2}}$

$(a_+ - a_-)(c_1 u_1 + c_2 u_2 + c_3 u_3) = c_1\sqrt{2}\,u_2 + c_2\sqrt{3}\,u_3 + c_3\sqrt{4}\,u_4 - c_1 u_0 - c_2\sqrt{2}\,u_1 - c_3\sqrt{3}\,u_2$

$= -c_1 u_0 - c_2\sqrt{2}\,u_1 + (c_1\sqrt{2} - c_3\sqrt{3})u_2 + c_2\sqrt{3}\,u_3 + c_3\sqrt{4}\,u_4$

$0 \qquad -c_2 c_1^*\sqrt{2} \qquad c_2^*(c_1\sqrt{2} - c_3\sqrt{3}) \qquad c_3^* c_2\sqrt{3} \qquad 0$

$\langle p \rangle = i\sqrt{\frac{\hbar m\omega}{2}} \left(-c_2 c_1^*\sqrt{2} + c_2^*(c_1\sqrt{2} - c_3\sqrt{3}) + c_3^* c_2\sqrt{3} \right)$

$\sqrt{2}\,(\underbrace{c_2^* c_1 - c_2 c_1^*}_{2i\,\mathrm{Im}(c_2^* c_1)}) \qquad \sqrt{3}\,(\underbrace{c_3^* c_2 - c_3 c_2^*}_{2i\,\mathrm{Im}(c_3^* c_2)})$

$2i\,\underbrace{\mathrm{Im}(c_2^* c_1)}_{2} \qquad 2i\,\mathrm{Im}(c_3^* c_2) = 0$

$= -4\sqrt{\frac{\hbar m\omega}{\sqrt{2}}}$

(6) $\psi = \frac{1}{\sqrt{2}}(u_1 + i u_2) \qquad \psi(x,t) = \frac{1}{\sqrt{2}}\left(u_1 e^{-\frac{iE_1 t}{\hbar}} + i u_2 e^{-iE_2 t/\hbar}\right)$

$= \frac{e^{-iE_2 t/\hbar}}{\sqrt{2}}\left(u_1 e^{i\omega t} + i u_2\right) \qquad \omega = \frac{E_2 - E_1}{\hbar}$

$\langle x \rangle = \frac{1}{2}\langle u_1 e^{i\omega t} + i u_2 | x | u_1 e^{i\omega t} + i u_2 \rangle$

$= \frac{1}{2}\Big\{ \underbrace{\langle u_1|x|u_1\rangle}_{L/2} + \underbrace{\langle u_1|x|u_2\rangle}_{\frac{-16}{9\pi^2}L} i e^{-i\omega t} + -i e^{i\omega t} \underbrace{\langle u_2|x|u_1\rangle}_{\frac{-16}{9\pi^2}L} + \underbrace{\langle u_2|x|u_2\rangle}_{L/2} \Big\}$

$= L/2 - \frac{16}{9\pi^2}L \sin(\omega t)$

same thing for p:

$\langle p \rangle = \frac{1}{2}\left\{ 0 + \frac{-8}{3}\frac{\hbar}{iL} i e^{-i\omega t} + -i e^{-i\omega t}\frac{8}{3}\frac{\hbar}{iL} + 0 \right\}$

$= -\frac{8}{3}\frac{\hbar}{L}\cos(\omega t) \qquad \omega = \frac{\hbar\pi^2}{2mL^2}(4-1)$

$\frac{16}{9\pi^2}L \cdot \omega = \frac{16}{9\pi^2}L\,\frac{\hbar\pi^2}{2mL^2}\,3 = \frac{8\hbar}{3mL}$ ✓