6.2 $E_n = \hbar\omega\sqrt{1+\varepsilon}\,(n+\tfrac{1}{2}) = \hbar\omega\left(1+\tfrac{1}{2}\varepsilon-\tfrac{1}{8}\varepsilon^2+\dots\right)(n+1/2)$

always "v_0"

$E^1 = \tfrac{1}{2}\hbar\omega(n+1/2)\,\varepsilon \qquad E^2 = -\tfrac{1}{8}\hbar\omega(n+1/2)\,\varepsilon^2$

$H = \frac{p^2}{2m} + \frac{1}{2}K x^2 = \left(\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2\right) + \underbrace{\left(\frac{1}{2}\varepsilon m\omega^2 x^2\right)}_{H'}$

$(1+\varepsilon)$

also $K = m\omega^2$

note $x = \sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-)$

$\langle m|H'|n\rangle = \frac{1}{2}\varepsilon m\omega^2 \frac{\hbar}{2m\omega}\langle m|(a_++a_-)^2|n\rangle \qquad x^2 = \frac{\hbar}{2m\omega}(a_++a_-)^2$

see $m = n+2,\ n,\ n-2$ or result is zero

$= \frac{1}{4}\varepsilon\hbar\omega\,\langle m|(a_++a_-)^2|n\rangle$

$a_+^2 \qquad a_+a_- + a_-a_+ \qquad a_-^2$

$a_+^2|n\rangle = a_+\sqrt{n+1}\,|n+1\rangle = \sqrt{n+1}\sqrt{n+2}\,|n+2\rangle$

$a_-^2|n\rangle = a_-\sqrt{n}\,|n-1\rangle = \sqrt{n}\sqrt{n-1}\,|n-2\rangle$

$(a_+a_-)|n\rangle = n|n\rangle \qquad a_-a_+|n\rangle = (n+1)|n\rangle$

so $\langle m|H'|n\rangle = \frac{1}{4}\varepsilon\hbar\omega \begin{cases} m=n+2 \to \sqrt{(n+1)(n+2)} \\ m=n \to 2n+1 \\ m=n-2 \to \sqrt{n(n-1)} \end{cases}$

First order: $\langle n|H'|n\rangle \to \frac{1}{2}\varepsilon\hbar\omega(n+1/2)$ ✓

Second order: $\frac{|\langle n+2|H'|n\rangle|^2}{E_n-E_{n+2}} + \frac{|\langle n-2|H'|n\rangle|^2}{E_n-E_{n-2}}$

$= \left(\frac{1}{4}\varepsilon\hbar\omega\right)^2\left\{\frac{(n+1)(n+2)}{-2\hbar\omega} + \frac{n(n-1)}{2\hbar\omega}\right\} \to \frac{-4n-2}{2\hbar\omega} = \frac{-2(n+1/2)}{\hbar\omega}$

$= -\frac{1}{8}\varepsilon^2\hbar\omega(n+1/2)$ ✓

6.3: single particle states $u_n = \sqrt{\frac{2}{L}}\sin(kx) \quad k = \frac{n\pi}{L} \quad E = \frac{\hbar^2k^2}{2m}$

gs: $u_1(x_1)\,u_1(x_2) \qquad E = \frac{\hbar^2\pi^2}{2mL^2}(1^2+1^2)$

excited: $\frac{1}{\sqrt{2}}\left(u_1(x_1)u_2(x_2) + u_2(x_1)u_1(x_2)\right) \qquad E = \frac{\hbar^2\pi^2}{2mL^2}(1^2+2^2)$

gs: $E^1 = \langle gs|H'|gs\rangle = -aV_0\int_0^L\int_0^L \left[u_1(x_1)u_1(x_2)\right]^2 \delta(x_1-x_2)\,dx_1dx_2$

$= -aV_0\int_0^L\left[u_1(x_1)u_1(x_1)\right]^2 dx_1$

$= -aV_0\left(\frac{2}{L}\right)^2\int_0^L \sin^4\left(\frac{\pi x}{L}\right)dx_1 = -aV_0\left(\frac{2}{L}\right)^2\frac{L}{\pi}\underbrace{\int_0^\pi \sin^4\theta\,d\theta}_{\frac{3}{8}\pi}$

$= -\frac{3}{2}\frac{aV_0}{L}$ oh - $a = L \to -\frac{3}{2}V_0$

6.3 excited: $-aV_0\int_0^L\int_0^L \psi^2\,\delta(x_1-x_2)\,dx_1dx_2 = -aV_0\int_0^L \psi^2(x_1,x_1)\,dx_1$

fn of x_1 & x_2

note $\psi(x_1,x_1) = \frac{1}{\sqrt{2}}\left(u_1(x_1)u_2(x_1)+u_2(x_1)u_1(x_1)\right) = \sqrt{2}\,u_1(x_1)u_2(x_1)$

so $E' = -aV_0\int_0^L 2\,u_1^2(x_1)u_2^2(x_1)\,dx_1 = -aV_0\,2\left(\frac{2}{L}\right)^2\int_0^L \sin^2\left(\frac{\pi x}{L}\right)\sin^2\left(\frac{2\pi x}{L}\right)dx$

$= -aV_0\,2\left(\frac{2}{L}\right)^2\frac{L}{\pi}\int_0^\pi \sin^2(\theta)\sin^2(2\theta)\,d\theta = -\frac{aV_0}{L}\,\mathrm{I}$

$\sin^2(\theta) \to \frac{1-\cos(2\theta)}{2}$, $\cos(2\theta) \to 0$, $\frac{\pi}{4}$

6.5 $\langle n|x|n\rangle = 0$ so $E' = 0$ $\quad x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$

$$\langle m|H'|n\rangle = -qE\langle m|x|n\rangle = -qE\begin{cases} m=n+1: & \sqrt{n+1}\sqrt{\frac{\hbar}{2m\omega}} \\ m=n-1: & \sqrt{n}\sqrt{\frac{\hbar}{2m\omega}}\end{cases}$$

zero unless
$m=n+1$
or
$m=n-1$

$$E' = q^2E^2\left\{\frac{n+1}{-\hbar\omega}+\frac{n}{\hbar\omega}\right\}\frac{\hbar}{2m\omega} = \frac{q^2E^2}{2\underbrace{m\omega^2}_{k}}(-1)$$

6.8 Let $u_n(x) \equiv \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$

gs: $u_1(x)u_1(y)u_1(z) \equiv \psi_0$ excited: $\begin{cases}\psi_x = u_2(x)u_1(y)u_1(z)\\ \psi_y = u_1(x)u_2(y)u_1(z)\\ \psi_z = u_1(x)u_1(y)u_2(z)\end{cases}$

$E' = \langle\psi_0|H'|\psi_0\rangle = a^3V_0\,u_1^2\left(\frac{a}{4}\right)u_1^2\left(\frac{a}{2}\right)u_1^2\left(\frac{3a}{4}\right)$

$= a^3V_0\left(\frac{2}{a}\right)^3\underbrace{\sin^2\left(\frac{\pi}{4}\right)}_{1/2}\underbrace{\sin^2\left(\frac{\pi}{2}\right)}_{1}\underbrace{\sin^2\left(\frac{3}{4}\pi\right)}_{1/2} = 2V_0$

seek 3x3 matrix -

$\langle\psi_x|H'|\psi_x\rangle = a^3V_0\left(\frac{2}{a}\right)^3\sin^2\left(\frac{\pi}{2}\right)\sin^2\left(\frac{\pi}{2}\right)\sin^2\left(\frac{3\pi}{4}\right) = 4V_0$

$\langle\psi_y|H'|\psi_y\rangle = 0$

$\langle\psi_z|H'|\psi_z\rangle = 4V_0$

$\langle\psi_x|H'|\psi_y\rangle = \langle\psi_z|H'|\psi_y\rangle = 0$

$\langle\psi_x|H'|\psi_z\rangle = a^3V_0\left(\frac{2}{a}\right)^3\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{2}\right)\sin^2\left(\frac{\pi}{2}\right)\sin\left(\frac{3\pi}{4}\right)\overset{-1}{\sin\left(\frac{3\pi}{2}\right)}$

$= -4V_0$

6.8 cont so $[H'] = 4V_0 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

eigen vectors → values $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \to 2 \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to 0 \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \to 0$

First order corrections: $8V_0, \underbrace{0, 0}$

degeneracy not broken

old exam #9

$\langle \psi_{12} | H' | \psi_{12} \rangle = \lambda \left(\frac{2}{L}\right)^2 \sin^2\left(\frac{\pi}{3}\right) \sin\left(\frac{\pi}{2}\right)^2 = \frac{\lambda}{L^2} 4 \cdot \frac{3}{4} = 3\frac{\lambda}{L^2}$

$\langle \psi_{21} | H' | \psi_{21} \rangle = \lambda \left(\frac{2}{L}\right)^2 \sin^2\left(\frac{2\pi}{3}\right) \sin\left(\frac{\pi}{4}\right)^2 = \frac{\lambda}{L^2} \frac{3}{2}$

$\langle \psi_{12} | H' | \psi_{21} \rangle = \lambda \left(\frac{2}{L}\right)^2 \underbrace{\sin\left(\frac{\pi}{3}\right) \sin\left(\frac{2\pi}{3}\right)}_{3/4} \underbrace{\sin\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi}{4}\right)}_{\frac{1}{\sqrt{2}}}$

$= \frac{\lambda}{L^2} \frac{3}{\sqrt{2}}$

$[H'] = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \frac{3\lambda}{L^2}$

$\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \leftarrow 0 \qquad 3/2 \to \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$

$\to (1-x)(\tfrac{1}{2}-x) - \tfrac{1}{2} = x^2 - \tfrac{3}{2}x = x\left(x - \tfrac{3}{2}\right) = 0$

$E' = 0 \;\; \text{ê} \;\; \frac{9}{2} \frac{\lambda}{L^2}$

↑4.5

The plots t304#9.pdf shows whats happening - this linear combo puts a node at the delta function for $E'=0$ and a near max at the delta function for $E' = 4.5\frac{\lambda}{L^2}$

[The red dot shows where the delta function is in these contour plots of $|\psi|^2$]

old exam 346t304 #5 $\quad \psi = u_E + \varepsilon\phi \quad$ where $H u_E = E u_E$

$$\langle\psi|H|\psi\rangle = \langle u_E|H|u_E\rangle + \varepsilon\langle u_E|H|\phi\rangle + \varepsilon\langle\phi|H|u_E\rangle + \varepsilon^2\langle\phi|H|\phi\rangle$$
$$+\varepsilon^2 E\langle\phi|\phi\rangle - \varepsilon^2\langle\phi|E|\phi\rangle$$
$$= E\left(\langle u_E|u_E\rangle + \varepsilon\langle u_E|\phi\rangle + \varepsilon\langle\phi|u_E\rangle + \varepsilon^2\langle\phi|\phi\rangle\right)$$
$$+\varepsilon^2\langle\phi|H-E|\phi\rangle$$

$\langle\psi|\psi\rangle$

$$= E\langle\psi|\psi\rangle + \varepsilon^2\langle\phi|H-E|\phi\rangle \checkmark$$

7.1 $\quad -\frac{\hbar^2}{2m}\partial_x^2 + \alpha|x|$

α : B units $\frac{E}{L}$

$\frac{\hbar^2}{2m}$: A units $E\cdot L^2$

$L \equiv \left(\frac{A}{B}\right)^{1/3}$

$E = (AB^2)^{1/3}$

$\rightarrow -\partial_x^2 + |x| = H$

$\psi = e^{-bx^2}$ $\qquad E = \dfrac{\langle\psi'|\psi'\rangle + \langle\psi|x|\psi\rangle}{\langle\psi|\psi\rangle} = \dfrac{\int_0^\infty \psi'^2 dx + \int_0^\infty x\psi^2 dx}{\int_0^\infty \psi^2 dx}$

math $\rightarrow E = \frac{3}{2\pi^{1/3}} \times \underbrace{\text{energy unit}}_{\left(\frac{\hbar^2}{2m}\alpha^2\right)^{1/3}}$

$b \,\dot{\mathrel{=}}\, \frac{1}{\sqrt{b}}$

$b = \frac{1}{2\pi^{1/3}} \times (\text{length unit})^{-2} = \frac{1}{2\pi^{1/3}}\left(\frac{2m\alpha}{\hbar^2}\right)^{2/3}$

$-\frac{\hbar^2}{2m}\partial_x^2 + \alpha x^4$

α : B: unit $\frac{E}{L^4}$

$\frac{\hbar^2}{2m}$: A: unit $E\cdot L^2$

$L = (A/B)^{1/6}$

$E = (A^2B)^{1/3}$

$-\partial_x^2 + x^4$

math $\rightarrow$ $E = \frac{3^{4/3}}{4} \times \underbrace{\text{energy unit}}_{\left(\frac{\hbar^4}{4m^2}\alpha\right)^{1/3}}$

$\frac{1}{b^2} \,\dot{\mathrel{=}}\, b$

$b = \frac{3^{1/3}}{2} \times \underbrace{(\text{length unit})^{-2}}_{\left(\frac{\hbar^2}{2m\alpha}\right)^{-1/3}}$

cosh #5

$u = sh\,x$

$du = ch\,x\,dx$

$\frac{du}{ch\,x} = dx$

$ch^2(x) = 1 + sh^2(x)$

$$WKB = \int_{tp_1}^{tp_2} \sqrt{E + \frac{V_0}{ch^2 x}}\, dx$$

$$= \int_{tp_1}^{tp_2} \frac{\sqrt{E\,ch^2 x + V_0}}{\cosh^2(x)}\, du$$

note $E < 0$ so write: $E = -|E|$

$$= \int \frac{\sqrt{E(1+sh^2)+V_0}}{1+sh^2}\, du$$

$$= \int \frac{\sqrt{-|E|(1+u^2)+V_0}}{1+u^2}\, du = \sqrt{|E|}\int \frac{\sqrt{\underbrace{\frac{V_0-|E|}{|E|}}_{A^2} - u^2}}{1+u^2}\, du$$

$$= \sqrt{|E|}\int_{-A}^{A} \frac{\sqrt{A^2-u^2}}{1+u^2}\, du = \sqrt{|E|}\,\pi\left(\sqrt{1+A^2}-1\right)$$

$$= \pi\left(\sqrt{|E| + V_0 - |E|} - \sqrt{|E|}\right) = \pi\left(\sqrt{V_0} - \sqrt{|E|}\right)$$

$$= \pi(n - 1/2)$$

so $\sqrt{|E|} = \sqrt{V_0} - (n - 1/2) \Rightarrow E = -\left(\sqrt{V_0} - (n-\tfrac{1}{2})\right)^2$

exact has $V_0 + \frac{1}{4}$ here otherwise same.

old exam #1 – actual wavefunction is plotted math/346+304#1.pdf

note: small amplitude & short wavelength near $x = \pm .7$ as that is where KE is greatest

note: for $n = 20$ 19 zeros & odd function.

#2 $$\int_{-E^{1/4}}^{E^{1/4}} \sqrt{E - x^4}\, dx \underset{\text{math}}{=} E^{3/4}\frac{\sqrt{\pi}\,\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{7}{4}\right)} = \pi(n - 1/2)$$

$$E = \left(\frac{\sqrt{\pi}(n-\frac{1}{2})\,\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}\right)^{4/3}$$

$$\psi'' = -\Big(\underbrace{\frac{V_0}{ch^2(x)} - |E|}_{k^2}\Big)\psi \qquad -|E| = +E \quad \text{as } E<0$$

$$u = sh(x) \qquad ch^2(x) - sh^2(x) = 1$$
$$du = ch(x)\,dx$$

$$\pi\big(\overbrace{n-\tfrac{1}{2}}^{\text{or } n+1/2}\big) = 2\int_0^{x_{max}} \sqrt{\frac{V_0}{ch^2(x)} - |E|}\,dx = 2\int \frac{\sqrt{V_0 - |E|ch^2(x)}}{ch(x)}\frac{du}{ch(x)}$$

$$= 2\int_0 \frac{\sqrt{V_0 - |E|(1+sh^2(x))}}{1+sh^2(x)}\,du = 2\int_0^{u_{max}} \frac{\sqrt{V_0 - |E| - |E|u^2}}{1+u^2}\,du$$

$$= 2\sqrt{|E|}\int_0^{u_{max}} \frac{\sqrt{\frac{V_0-|E|}{|E|} - u^2}}{1+u^2}\,du$$

$$= 2\sqrt{|E|}\,\frac{\pi}{2}\left(\sqrt{1 + \frac{V_0-|E|}{|E|}} - 1\right) = \pi\left(\sqrt{V_0} - \sqrt{|E|}\right)$$

so $$\sqrt{|E|} = \sqrt{V_0} - (n+1/2)$$

$$E = -\left(\sqrt{V_0} - (n+1/2)\right)^2$$

$$\sqrt{\pi}\,\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{7}{4})} = .874 \equiv \alpha$$

$$\psi'' = -(\underbrace{E - x^4}_{k^2})\psi \qquad \text{turning pts at } \pm E^{1/4}$$

$$\pi(n-1/2) = 2\int_0^{E^{1/4}} \sqrt{E - x^4}\,dx = 2E^{3/4}\overbrace{\int_0^1 \sqrt{1-u^4}\,du}$$

$$\cancel{x} = E^{1/4}u$$

$$\frac{\sqrt{\pi}(n-1/2)}{2\alpha} = E^{3/4} \Rightarrow \quad E = \left(\frac{\sqrt{\pi}(n-1/2)}{2\alpha}\right)^{4/3}$$

tdpt eq 9.17 $c_b^{(1)} = \frac{-i}{\hbar}\int_0^t H'_{ba}\, e^{i\omega_0 t'}\, dt'$

$0 \to -\infty$ $\quad H'_{ba} \to A_{ba}\,\frac{e^{-t^2/\tau^2}}{\sqrt{\pi}\,\tau}$

$$c_m = \frac{-i}{\hbar} A_{m0} \frac{1}{\sqrt{\pi}\,\tau} \underbrace{\int_{-\infty}^{\infty} e^{-t^2/\tau^2 + i\omega_0 t}\, dt}$$

compare cheat sheet integral. $\int_{-\infty}^{\infty} e^{-\alpha x^2 - \beta x}\, dx = \sqrt{\frac{\pi}{\alpha}}\, e^{\beta^2/}$

$$\sqrt{\pi \tau^2}\, e^{-\omega_0^2/4\,1/\tau^2} = \sqrt{\pi}\,\tau\, e^{-\omega_0^2\tau^2/4}$$

$$c_m = \frac{-i}{\hbar} A_{m0}\, e^{-\omega_0^2\tau^2/4}$$ Note $\omega_0 = \frac{E_m - E_0}{\hbar}$

e^{-x} goes to zero if $x \to \infty$; here $x = \frac{\omega_0^2\tau^2}{4}$

e^{-x} goes to one if $x \to 0$ $\Rightarrow c_m = \frac{-i}{\hbar} A_{m0}$

$$\sum_{m\neq 0} |c_m|^2 = \frac{1}{\hbar^2}\sum_{m\neq 0} A_{0m} \underset{A^*_{m0}}{A_{m0}} = \frac{1}{\hbar^2}\left(\sum_{\text{all}} \underbrace{A_{0m}A_{m0}} - A_{00}A_{00}\right)$$

$$= \frac{1}{\hbar^2}\left(\langle 0|A^2|0\rangle - \langle 0|A|0\rangle^2\right)$$

$\langle 0|A\underbrace{|m\rangle\langle m|}_{=1}A|0\rangle$

$\langle 0|AA|0\rangle = \langle 0|A^2|0\rangle$

Morse

$$V = D\left(1 - e^{-\frac{(r-r_e)}{a}}\right)^2$$

$$= D\left(1 - e^{-x/a}\right)^2 \simeq \frac{D}{a^2} x^2 \left(1 - \frac{x}{a} + \frac{7}{12}\left(\frac{x}{a}\right)^2 \right.$$

ie: $\frac{D}{a^2} = \frac{1}{2} m\omega^2$

$$\underbrace{-\frac{\hbar^2}{2m}}\partial_x^2 \psi + \frac{1}{2} m\omega^2 a^2 \left(1 - e^{-x(a)}\right)^2 \psi = E\psi$$

$E \leftarrow E'\left(\frac{1}{2} m\omega^2 a^2\right)$

$$\frac{\frac{\hbar^2}{2ma^2}}{\frac{1}{2} m\omega^2 a^2} = \frac{\hbar^2}{m^2\omega^2 a^4} \equiv \frac{1}{\alpha^2}$$

$$\psi'' = -\alpha^2\left(E' - (1 - e^{-x})^2\right)\psi$$

$$\pi(n + 1/2) = \alpha \int \sqrt{\underset{a^2}{E'} - (1 - \underbrace{e^{-x}}_{u})^2}\, dx = \alpha \int_{-a}^{a} \frac{\sqrt{a^2 - u^2}}{1 - u}\, du$$

$0 \to 1$

$E' = a^2$

$du = e^{-x}\, dx = (1 - u)\, dx$

$-i(2\pi i)$

$$2\int_{-a}^{a} \frac{\sqrt{a^2 - u^2}}{1 - u} = i \oint \frac{\sqrt{a^2 - u^2}}{1 - u} = i\left\{ +2\pi i \sqrt{1 - a^2} + \oint \frac{-ix\sqrt{1 - \frac{a^2}{x^2}}}{x - 1} \right\}$$

$\downarrow$

$x\left(1 - \frac{1}{2}\frac{a^2}{x^2}\right)\left(1 + \frac{1}{x}\right)$

$-\frac{1}{x}$

$$= 2\pi \left\{ -\sqrt{1 - a^2} + 1 \right\}$$

$$\left(n + \tfrac{1}{2}\right) = \alpha \left\{ 1 - \sqrt{1 - a^2} \right\}$$

$$\sqrt{1 - a^2} = 1 - \frac{(n + \frac{1}{2})}{\alpha}$$

$$1 - \left(1 - \frac{(n + \frac{1}{2})}{\alpha}\right)^2 = a^2 = \frac{2(n + \frac{1}{2})}{\alpha} - \frac{(n + \frac{1}{2})^2}{\alpha^2} = E \Big/ \left(\tfrac{1}{2} m\omega^2 a^2\right)$$

$$E = \underbrace{\frac{m\omega^2 a^2}{\alpha}}_{\hbar\omega}(n + 1/2) - \underbrace{\frac{\frac{1}{2} m\omega^2 a^2}{m^2\omega^2 a^4 / \hbar^2}}_{\frac{\hbar^2}{2ma^2}}\left(n + \tfrac{1}{2}\right)^2$$

morse #3

$$\frac{-\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega^2\left[x^2 - \frac{x^3}{2a} + \frac{7}{12}\frac{x^4}{a^2} + \cdots\right]$$

$\langle x^3\rangle = 0$ as x^3 is odd but ψ^2 is even

$$H' = \frac{1}{2}m\omega^2 \frac{7}{12a^2}\langle n|x^4|n\rangle$$

$$\hookrightarrow x = \sqrt{\frac{\hbar}{2m\omega}}\left(a_+ + a^-\right)$$

$$= \frac{1}{2}m\omega^2\frac{7}{12a^2}\left(\frac{\hbar}{2m\omega}\right)^2 \langle n|(a_+ + a_-)^4|n\rangle$$

$a_+a_+a_-a_- \rightarrow \sqrt{n}\sqrt{n-1}\sqrt{n-1}\sqrt{n} = n(n-1)$

$a_+a_-a_+a_- \rightarrow \sqrt{n}\sqrt{n}\sqrt{n}\sqrt{n} = n^2$

$a_-a_+a_+a_- \rightarrow \sqrt{n+1}\sqrt{n+1}\sqrt{n}\sqrt{n} = n(n+1)$

$a_+a_-a_-a_+ \rightarrow \sqrt{n}\sqrt{n}\sqrt{n+1}\sqrt{n+1} = n(n+1)$

$a_-a_+a_-a_+ \rightarrow \sqrt{n+1}\sqrt{n+1}\sqrt{n+1}\sqrt{n+1} = (n+1)^2$

$a_-a_-a_+a_+ \rightarrow \sqrt{n+1}\sqrt{n+2}\sqrt{n+2}\sqrt{n+1} = (n+2)(n$

$6n^2 + 6n + 3$

$$= \frac{7}{48}\frac{\hbar^2}{2ma^2}\left(6n^2+6n+3\right)$$

$$= \frac{7}{32}\frac{\hbar}{ma^2}\left(2n^2+2n+1\right)$$

$H2$ of $\quad -\frac{1}{2} m\omega^2 \frac{x^3}{a}$

units: $\dfrac{\left(\frac{1}{2} m\omega^2 \frac{1}{a} \left(\frac{\hbar}{2m\omega}\right)^{3/2}\right)^2}{\hbar\omega} = \dfrac{\hbar^2}{16\, m\, a^2}$

$\langle m | (a_+ + a_-)^3 | n\rangle \xrightarrow{m=n+3} \sqrt{n+3}\,\sqrt{n+2}\,\sqrt{n+1} \rightarrow \dfrac{(n+1)(n+2)(n+3)}{-3}$

$\xrightarrow{m=n-3} \sqrt{n-2}\,\sqrt{n-1}\,\sqrt{n} \rightarrow \dfrac{(n-2)(n-1)\,n}{3}$

$3n^2+3n+2$

$m = n+1 \quad a_+a_+a_- \quad \sqrt{n+1}\,\sqrt{n}\,\sqrt{n} \rightarrow n^2(n+1)$

$a_+a_-a_+ \quad \sqrt{n+1}\,\sqrt{n+1}\,\sqrt{n+1} \rightarrow (n+1)^3$

$a_-a_+a_+ \quad \sqrt{n+2}\,\sqrt{n+2}\,\sqrt{n+1} \rightarrow (n+2)^2(n+1)$

$\Rightarrow \dfrac{3n^3 + 9n^2 + 11n + 5}{-1}$

$m = n-1 \quad a_+a_-a_- \quad \sqrt{n-1}\,\sqrt{n-1}\,\sqrt{n} \rightarrow (n-1)^2 n$

$a_-a_+a_- \quad \sqrt{n}\,\sqrt{n}\,\sqrt{n} \rightarrow n^3$

$a_-a_-a_+ \quad \sqrt{n}\,\sqrt{n+1}\,\sqrt{n+1} \rightarrow n(n+1)^2$

$9n^2+9n+5$

$\dfrac{3n^3 + 0n^2 + 2n + 0}{1}$

$\frac{1}{8}(12n^2+12n+9) - \frac{7}{48}(6n^2+6n+3) = \frac{5}{8}(n^2+n) + \frac{3.5}{8}$ $\qquad \overline{12n^2+12n+7}$