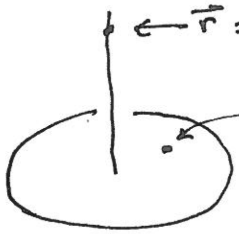
341 class 4 2-5a, 2-15a, old exam #3

2-5a



$\leftarrow \vec{r} = (0,0,z)$

$\vec{r}\,' = (r'\cos\phi, r'\sin\phi, 0)$

$\overrightarrow{r-r'} = (-r'\cos\phi, -r'\sin\phi, z)$

$|\vec{r}-\vec{r}\,'| = \sqrt{r'^2+z^2}$

$dQ = \sigma\, r'\,dr'\,d\phi$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_0^R\int_{-\pi}^{\pi} \frac{(-r'\cos\phi, -r'\sin\phi, z)}{(r'^2+z^2)^{3/2}}\,\sigma\, r'\,dr'\,d\phi$$

For E_x & E_y : $\int_{-\pi}^{\pi}\cos\phi\, d\phi = \int_{-\pi}^{\pi}\sin\phi\, d\phi = 0$; $\int_{-\pi}^{\pi} d\phi = 2\pi \leftarrow E_z$

$$E_z = \frac{\sigma 2\pi}{4\pi\varepsilon_0}\int_0^R \frac{z}{(r'^2+z^2)^{3/2}}\,r'\,dr'$$ substitute $u = r'^2$, $du = 2r'dr$

$$= \frac{\sigma\pi}{4\pi\varepsilon_0} z \underbrace{\int_0^R (u+z^2)^{-3/2}\,du}_{\left.\frac{(u+z^2)^{-1/2}}{-1/2}\right|_0^R \;=\; 2\left(\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right)}$$

$$= \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)$$

Note: $z \to 0 \Rightarrow \frac{\sigma}{2\varepsilon_0}$

$z \to \infty$ $\left(1 - \underbrace{\frac{1}{\sqrt{1+R^2/z^2}}}_{1-\frac{1}{2}\frac{R^2}{z^2}}\right)$

$= \frac{1}{2}\frac{R^2}{z^2}$

$(1+\varepsilon)^{\alpha} \approx 1 + \alpha\varepsilon$, $\alpha = -1/2$, $\varepsilon = \frac{R^2}{z^2}$

Overall

$$\frac{\sigma}{2\varepsilon_0}\frac{1}{2}\frac{R^2}{z^2} = \frac{\sigma\pi R^2}{4\pi\varepsilon_0 z^2} = \frac{Q}{4\pi\varepsilon_0 z^2}$$

2-15a) $\rho = \frac{A}{r}$ total charge $= \int_0^R \rho\, 4\pi r^2 dr = 4\pi A \int_0^R r\, dr$

$= 4\pi A \frac{1}{2} R^2 = 2\pi A R^2$

so for $r > R$ $E_r = \frac{Q}{4\pi\varepsilon_0 r^2}$

$= \frac{2\pi A R^2}{4\pi \varepsilon_0 r^2} = \frac{A}{2\varepsilon_0}\left(\frac{R}{r}\right)^2 = -\partial_r \phi(r)$

so $\phi(r) = \frac{AR^2}{2\varepsilon_0}\frac{1}{r}$

Note: $\phi(a) - \phi(b) = \int_a^b -\vec{\nabla}\phi \cdot d\vec{r} = \int_a^b \vec{E}\cdot d\vec{r}$

if $b = \infty$ & $\phi(b) = 0$ $\phi(a) = \int_a^\infty \vec{E}\cdot dr = \int_a^\infty \frac{AR^2}{2\varepsilon_0 r^2} dr$

Note: $\phi(R) = \frac{AR}{2\varepsilon_0}$ $= \frac{AR^2}{2\varepsilon_0}\left.\frac{r^{-1}}{-1}\right|_a^\infty = \frac{AR^2}{2\varepsilon_0}\frac{1}{a}$ ✓

for $r < R$ need Q inside $r = \int_0^r \rho\, 4\pi r^2 dr = 2\pi A r^2$

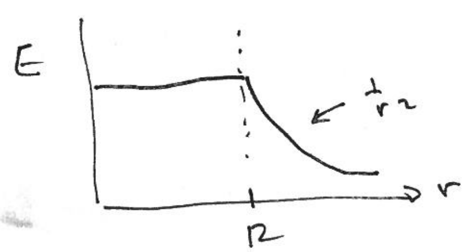
so $E_r = \frac{1}{4\pi\varepsilon_0}\frac{2\pi A r^2}{r^2} = \frac{A}{2\varepsilon_0}$ ← constant

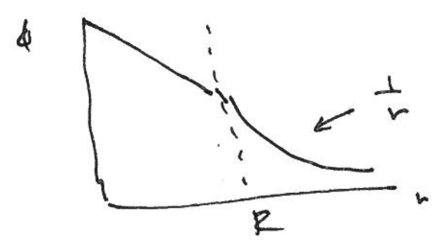
$\phi(a) - \phi(b) = \int_a^b \vec{E}\cdot dr = \int_a^b \frac{A}{2\varepsilon_0} dr = \frac{A}{2\varepsilon_0}(b-a)$

$b = R$

$\phi(b) = \frac{AR}{2\varepsilon_0}$

$\phi(a) = \frac{AR}{2\varepsilon_0} + \frac{A}{2\varepsilon_0}(\overset{R}{b} - a) = \frac{A}{2\varepsilon_0}(2R - \overset{r}{a})$





old exam #3

→ for voltage — no integral required — all charge is at same distance for origin (R) so $\phi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R}$

when $Q = \lambda \pi R$

→ for electric field:

$$\vec{r} = (0,0,0)$$
$$\vec{r}\,' = (R\cos\theta, R\sin\theta, 0)$$
$$\overrightarrow{r-r'} = (-R\cos\theta, -R\sin\theta, 0)$$
$$|\vec{r}-\vec{r}\,'| = R$$
$$dQ = \lambda R\, d\theta$$

$$\bar{E} = \frac{1}{4\pi\varepsilon_0}\int_{-\pi/2}^{\pi/2} \frac{(-R\cos\theta, -R\sin\theta, 0)}{R^3}\,\underbrace{\lambda R}_{Q/\pi}\, d\theta$$

$$= \frac{-1}{4\pi\varepsilon_0}\frac{\lambda}{R}\int_{-\pi/2}^{\pi/2} (\cos\theta, \sin\theta, 0)\, d\theta$$

$$\cos\theta \to \sin\theta \qquad \sin\theta \to -\cos\theta\Big|_{-\pi/2}^{\pi/2} = 0$$

$$\sin\theta\Big|_{-\pi/2}^{\pi/2} = 2$$

$$= \frac{-1}{4\pi\varepsilon_0}\frac{\lambda}{R}(2, 0, 0)$$ ← no surprise by symmetry $E_y = 0$