Multipole Expansion Example

The multipole expansion is useful both in theory and in practice. In theory we learn that, viewed from a distance, the electric field from any object will approximately be that of the first non-zero multipole. In practice, calculating the multipoles is much easier than directly calculating the electric field because the required integrals just involve the charge distribution of the object (and not the location of the observation point), but the resulting expansion gives us (approximately) the fields at any observation point 'far' from the object.

Consider a not-too-simple simple charge distribution: a uniformly charged ring of radius R with center at the origin that sits in the xy plane. Since we're going to be doing a good bit of algebra, simple expressions are helpful. In this particular problem everywhere there would be an overall factor of

$$\frac{\lambda}{4\pi\epsilon_0}$$

where λ is the charge per length of the ring. We will ignore this overall factor. Without loss of generality we seek the voltage at the 'arbitrary' point $\mathbf{r} = (x, 0, z)$ by integrating over the source (the charged ring) $\mathbf{r}' = (R\cos\phi, R\sin\phi, 0)$:

$$\phi = \int \frac{R\,d\phi}{\sqrt{x^2 + R^2 - 2Rx\cos\phi + z^2}}$$

We will eventually do this integral, but if we couldn't we could always approximate the field using the multipole expansion.

The monopole term (total charge) is easy: $\lambda 2\pi R$; having pulled out the above overall factor that term gives:

$$\phi = \frac{2\pi R}{r}$$

There is no dipole term as the center of charge is the origin.

So the fun starts with the quadrupole term…we just need to integrate over the source distribution:

$$Q_{ij} = \int \left(3x'_i x'_j - r'^2\delta_{ij}\right)\lambda R\,d\phi$$

where $\mathbf{r}' = (R\cos\phi, R\sin\phi, 0)$.

The $x'_i x'_j$ part of this 3×3 matrix is the dyadic $\mathbf{r}'\mathbf{r}'$ (the notation is two vectors sitting next to each other with no operation between them); this is also known as an outer product.

$$\mathbf{r}'\mathbf{r}' = \begin{pmatrix} x'x' & x'y' & x'z' \\ y'x' & y'y' & y'z' \\ z'x' & z'y' & z'z' \end{pmatrix}$$

The δ_{ij} is the identity matrix. In *Mathematica* we write:

```
Q=3 Outer[Times,{x,y,z},{x,y,z}]-(x^2+y^2+z^2) IdentityMatrix[3]
```

or

$$Q = \begin{pmatrix} 2x^2 - y^2 - z^2 & 3xy & 3xz \\ 3xy & -x^2 + 2y^2 - z^2 & 3yz \\ 3xz & 3yz & -x^2 - y^2 + 2z^2 \end{pmatrix}$$

We need to plug in our particular $\mathbf{r}'$ and do the resulting integral

```
Q /. {x->R Cos[p],y->R Sin[p],z->0}
Integrate[% R,{p,-Pi,Pi}]
Out[3]= R^3 {{Pi, 0, 0}, {0, Pi, 0}, {0, 0, -2 Pi}}
```

$$Q = \pi R^3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Since our axes are aligned with the symmetry of the object, Q is diagonal, and of course $\mathrm{Tr}(Q) = 0$.

The multipole expansion then says the resulting term in the electric potential is

$$\phi = \frac{\hat{\mathbf{r}} \cdot Q \cdot \hat{\mathbf{r}}}{2r^3}$$

For our field observation point $\hat{\mathbf{r}} = (\sin\theta, 0, \cos\theta)$:

```
{Sin[t],0,Cos[t]}.%3.{Sin[t],0,Cos[t]}/(2 r^3)
Simplify[%]
        -(Pi (1 + 3 Cos[2 t])) R^3
Out[5]= ----------------------
                     3
                  4 r
```

```
phi2=% + 2 Pi/r
```

While we are here I want to re-write the angular term in the way that will be more common in the future:

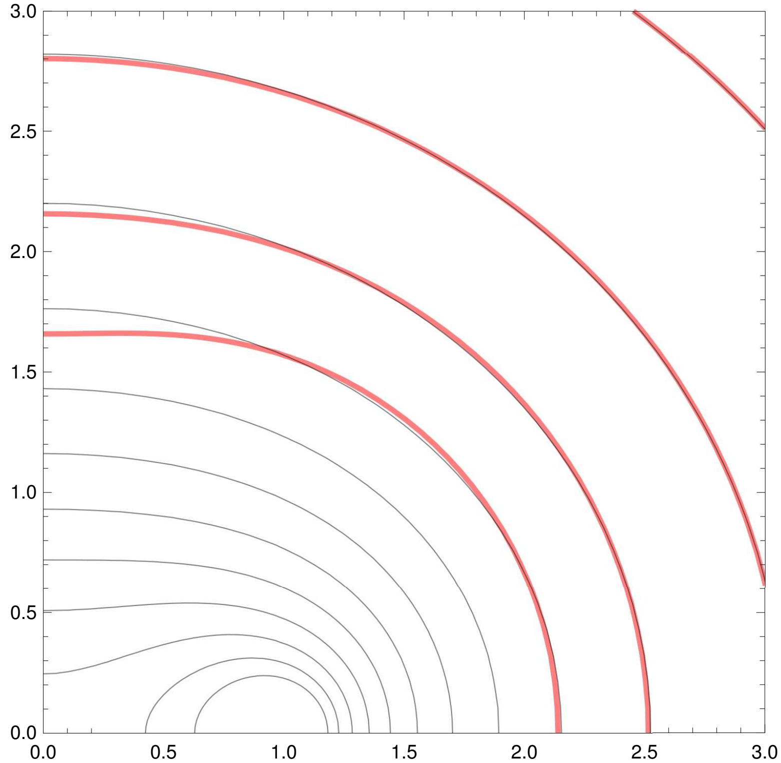
$$\frac{1}{4}\left(1 + 3\cos(2\theta)\right) = \frac{1}{2}\left(3\cos^2\theta - 1\right) = L_2(\cos\theta)$$

Now actually *Mathematica* can directly calculate the integral:

```
Integrate[R/Sqrt[x^2+z^2+R^2-2 x R Cos[p]],{p,-Pi,Pi},Assumptions-> x>0&&R>0&&z>0]

                              4 R x
         4 R EllipticK[-------------]
                                 2    2
                        (R + x)  + z
Out[7]=  ----------------------------
                             2    2
             Sqrt[(R + x)  + z ]

phi=Simplify[% /. {z-> r Cos[t],x->r Sin[t]} ]

                               4 r R Sin[t]
         4 R EllipticK[----------------------]
                             2    2
                        r  + R  + 2 r R Sin[t]
Out[8]=  -------------------------------------
                        2    2
             Sqrt[r  + R  + 2 r R Sin[t]]

Series[phi,{r,Infinity,5}]
Simplify[%]

                         3
          2 Pi R   Pi R  (1 + 3 Cos[2 t])
Out[10]=  ------ - --------------------- +
            r                   3
                             4 r

           5
     3 Pi R  (9 + 20 Cos[2 t] + 35 Cos[4 t])        -6
>    --------------------------------------- + O[r]
                                 5
                          256 r
```

The first two terms match the monopole and quadrupole terms we found directly from the charge distribution, the third would be the 16-pole term.

We can compare the approximate voltage (red) to the exact result:



It seems for $r > 2R$ the approximation closely follows the exact result.

It may be worth noting that we can also do a multipole expansion in the reverse case of distant *sources*: $r \ll r'$. For this ring problem the result is:

$$\phi = 2\pi - \pi L_2(\cos\theta)\;(r/R)^2$$