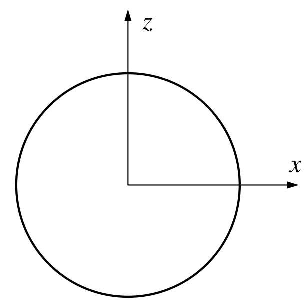


1. (24 pts) A plastic sphere of radius R with dielectric constant K is placed in what was a uniform electric field in the z direction: $E_0\widehat{\mathbf{k}}$. Aim: find the resulting voltage ϕ inside and outside the sphere. This is a rather long problem, so let's step through the problem together. What counts in this problem is explanations (words) not calculations.

   (a) The provided circle represents the pole-to-pole great circle that is the intersection of the xz plane and the sphere. It turns out that the polarization inside the sphere is uniform. Draw this diagram on your answer sheet. Draw the arrow showing the direction of the polarization. Mark the location of any bound surface charges: mark '+' for positive surface charge, and '−' for negative. Would you call this an even, odd or neither surface charge distribution? Explain.

   (b) I claim the following general forms for the voltage inside ($r < R$) and outside ($r > R$) the sphere:

$$\begin{aligned}\phi_{\text{in}}(r,\theta) &= \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)\\ \phi_{\text{out}}(r,\theta) &= \sum_{n=0}^{\infty} C_n r^{-n-1} P_n(\cos\theta) - E_0 r P_1(\cos\theta)\end{aligned}$$

   where A_n and C_n are currently undetermined constants. Explain (words) why ϕ must have this particular form. What assumptions have been used?

   (c) At the boundary (i.e., $r = R$) the following conditions must hold:

$$\begin{aligned}\phi_{\text{in}}(R,\theta) &= \phi_{\text{out}}(R,\theta)\\ K\left.\frac{\partial\phi_{\text{in}}}{\partial r}\right|_{r=R} &= \left.\frac{\partial\phi_{\text{out}}}{\partial r}\right|_{r=R}\end{aligned}$$

   Explain (words) why these particular conditions must hold.

   (d) I conclude from the first condition that

$$\begin{aligned}A_n &= C_n/R^{2n+1} && \text{for: } n \neq 1\\ A_1 &= C_1/R^3 - E_0 && \text{for: } n = 1\end{aligned}$$

   Explain the basis for this conclusion. How does one equation produce an infinite number of equations: one for every whole number n?

   (e) The second boundary condition yields

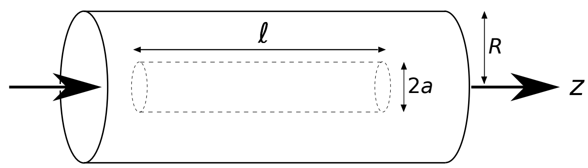
$$\begin{aligned}KnA_n &= -(n+1)C_n/R^{2n+1} && \text{for: } n \neq 1\\ KA_1 &= -2C_1/R^3 - E_0 && \text{for: } n = 1\end{aligned}$$

   It is easily check that a solution (for: $n \neq 1$) is $A_n = C_n = 0$. Explain how (words) that result means that the polarization inside the sphere is constant.

2. (24 pts) This is a long problem that is based on #1 from Exam 3 (so you might want to go back and look at it). I 'simplify' it a bit by making the wire non-magnetic (but all this really does is $\mu \to \mu_0$). SO:

A long straight cylindrical wire made out of non-magnetic conducting material (e.g., Cu) has radius R and carries a steady current I in the positive z direction uniformly distributed throughout its cross-section (so the current density $J = I/\pi R^2$ is uniform). The wire is the core of a coaxial cable with its current return being a zero-thickness conducting shell on the outer surface of the core but insulated from it. Inside ($r < R$) the core $\mathbf{B}$ and $\mathbf{A}$ are given by (in cylindrical coordinates)

$$\begin{aligned} \mathbf{B} &= \mu_0 \, \frac{rJ}{2} \, \widehat{\phi} \\ \mathbf{A} &= \mu_0 \, \frac{R^2 - r^2}{4} \, J \, \widehat{\mathbf{k}} \end{aligned}$$



and outside ($r > R$) the core both $\mathbf{B}$ and $\mathbf{A}$ are zero. Inside of the conductor, imagine a coaxial cylindrical volume (radius a, length ℓ). Problem: show that the energy production inside this cylinder (essentially inverse Joule heating) equals the rate of magnetic energy growth inside the cylinder plus the outward flow of energy (Poynting vector) though the cylindrical boundary $r = a$.

(a) Using Ampère's Law derive the formula for $\mathbf{B}$ ($r < R$) given above. (For an amperian loop $r > R$, the enclosed current is zero (the current up the core matches the down current on the outer boundary) and hence B is zero outside.)

(b) As in Eq. (16-60) the electric field is given by

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = -\mu_0 \, \frac{R^2 - r^2}{4} \, \dot{J} \, \widehat{\mathbf{k}}$$

where $\dot{J}$ is the time derivative of J. Using cylindrical coordinates take the curl of this $\mathbf{E}$ and verify that:

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

(c) Note that $\mathbf{J}$ and $\mathbf{E}$ are in opposite directions—that's why $(-\mathbf{J} \cdot \mathbf{E})$ is positive (energy production) rather than negative (Joule heating). Calculate the total energy production inside the inner cylinder (i.e., $r < a$) by

$$\int (-\mathbf{J} \cdot \mathbf{E}) \, dV$$

As in #1 Exam 3, the volume element is $dV = 2\pi r \; dr \; \ell$ and you should again explain (words) why this is the proper volume element.

(d) Calculate the total magnetic energy inside the cylinder by

$$\frac{1}{2\mu_0} \int B^2 \, dV$$

and take its time derivative to get the rate of increase in this energy.

(e) Calculate the flow of energy out of the cylinder by integrating the Poynting vector over the surface of the cylinder,

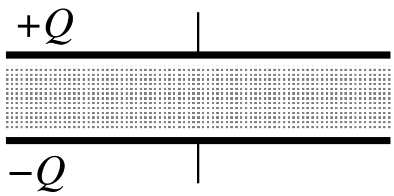
(f) Finally show that these three quantities add up so that energy production inside this cylinder equals the rate of magnetic energy growth inside the cylinder plus the outward flow of energy though the cylindrical boundary at $r = a$.

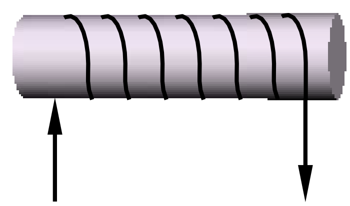
3. Consider a coaxial capacitor (length L, inner conductor radius a, outer b, filled with dielectric with permittivity ϵ) so long that we can approximate it as an infinitely long coaxial capacitor. The inner conductor carries total free charge Q; the outer carries an opposite free charge so the total free charge is zero.

   (a) Using Gauss's Law, find $\mathbf{D}$ and then $\mathbf{E}$ for $b > r > a$.

   (b) Find the potential difference between the coaxial cylinders.

   (c) Find the capacitance of the capacitor.



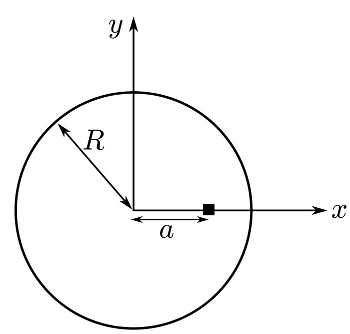


4. The below two questions request descriptions (drawings, labels & words, not calculations) of how materials affect electrical and magnetic properties.

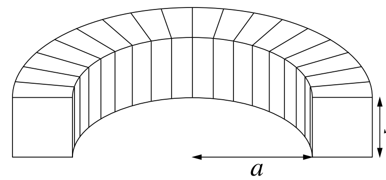
   (a) Consider linear dielectric material located between the plates of a capacitor as in the above left diagram. Draw your own version of the above. Draw and label a typical polarization vector $\mathbf{P}$ in the dielectric. Locate (draw arrows on the diagram showing location and sign) the bound charges. The volume bound charge density ρ_B is zero in this situation. Describe why/how the dielectric reduces the electric field (for fixed charge) and why that increases the capacitance. Note: quoting an equation like $C = K\epsilon_0 A/d$ is not an explanation

   (b) Consider linear (large K_m) material located in a solenoid as shown in the above right diagram. Draw your own version of the above. Draw and label a typical magnetization vector $\mathbf{M}$. Locate (draw arrows on the diagram showing location and flow direction) of the bound surface currents $\mathbf{j_M}$. The volume bound current density $\mathbf{J}_M$ is zero in this situation.The large K_m material fill increases the $\mathbf{B}$ inside the solenoid compared to an equivalent air-filled solenoid (for the same current); explain why/how.

5. A circle (radius R) sits in the xy plane and is made of a wire with uniform charge per length λ. Find the electric field at an arbitrary point $(a, 0)$ on the x axis. Draw a diagram clearly showing exactly what you are using for $\mathbf{r}$, $\mathbf{r}'$ and dq. Carefully set up the integral to find the electric field vector. (*Mathematica* can do the integral, but it requires some knowledge of elliptical integrals and is not part of this question.)



6. A toroid has a square cross section (side s), an inner radius of a, and a total of N turns each carrying a current I. (The diagram to the left shows half of this toroid.)



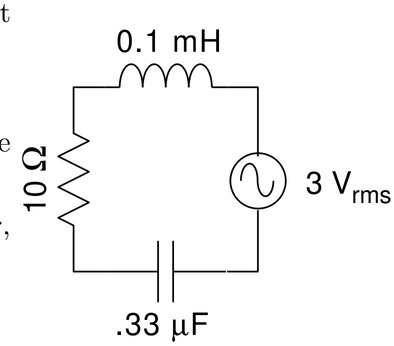
   (a) Using Ampère's Law find the magnetic field inside the toroid as a function of the distance from the axis of the toroid.

   (b) By integration, show that the magnetic flux through a single square turn around the torus is given by

$$\Phi_B = \frac{\mu_0 N s I}{2\pi} \ln\left(\frac{a+s}{a}\right)$$

   (c) Find the (self) inductance L of this toroid.

7. Consider the below right LRC circuit, driven by a generator at a frequency of $f = 25$ kHz. FYI: feel free to use your calculator in complex number mode to answer these questions, but record sufficient steps to allow me to follow your method.



   (a) Report the circuit impedance Z as a complex number. Report its magnitude and phase.

   (b) What rms current flows through the circuit?

   (c) What is the rms voltage across the inductor (V_L)? Across the capacitor (V_C)? Across the resistor (V_R)?

   (d) Report the power dissipated in the (i) resistor, (ii) inductor, (iii) capacitor.

   (e) Circle one: the total voltage leads the current
   the total voltage lags the current

Please sign the following statement:

In answering these questions I have used no aids other then the textbook and my personal notes.