1. (20 points) A long straight cylindrical wire made out of linear magnetic material ($\chi_m \gg 1$, e.g., iron rather than copper), has radius R and carries a steady current I uniformly distributed throughout its cross-section (so the current density $J = I/\pi R^2$ is uniform).

   The wire is the core of a coaxial cable with its current return being a zero-thickness insulated copper shell coaxial to and on the outer surface of the cylindrical core of magnetic material. An application of Ampere's Law should convince you that inside ($r < R$) the core $\mathbf{H}$ and $\mathbf{A}$ are given by (in cylindrical coordinates)

$$\begin{aligned} \mathbf{H} &= \frac{rJ}{2}\,\widehat{\boldsymbol{\phi}} \\ \mathbf{A} &= \mu\,\frac{R^2 - r^2}{4}\,J\,\widehat{\mathbf{k}} \end{aligned}$$

   and outside ($r > R$) the core both $\mathbf{H}$ and $\mathbf{A}$ are zero.

   (a) Calculate $\boldsymbol{\nabla} \times \mathbf{A}$ (in cylindrical coordinates) and confirm that the resulting $\mathbf{B}$ agrees with the formula I've given for $\mathbf{H}$. (Remark: for cylindrical coordinates what the textbook calls θ, I call ϕ.)

   (b) The effect of the magnetic material is of course to increase the resulting $\mathbf{B}$ over what it would be with non-magnetic material. We can explain the boosted B in terms of additional, mathematically-equivalent atomic magnetization currents (surface & volume)

$$\begin{aligned} \mathbf{j}_M &= \mathbf{M} \times \widehat{\mathbf{n}} \\ \mathbf{J}_M &= \boldsymbol{\nabla} \times \mathbf{M} \end{aligned}$$

   Calculate these quantities. Describe (words) where these currents are located and which way they are flowing; finally explain why (words) these currents would act to increase $\mathbf{B}$ inside the core and have no net effect outside the core.

   (c) The mathematically equivalent magnetic monopole density (surface & volume) should also provide an explanation of the increased $\mathbf{B}$

$$\begin{aligned} \sigma_M &= \mathbf{M} \cdot \widehat{\mathbf{n}} \\ \rho_M &= -\boldsymbol{\nabla} \cdot \mathbf{M} \end{aligned}$$

   Calculate these quantities; they should both turn out to be zero (which means they can't explain much—a puzzle we're not going to explore)

   (d) It should be possible to calculate the magnetic energy inside a length ℓ of the core via:

$$U = \frac{1}{2} \int \mathbf{H} \cdot \mathbf{B}\, dV$$

   where the volume element is $dV = 2\pi r\; dr\; \ell$. Explain (words) why this is the proper volume element and calculate the energy U.

   (e) Calculate also the magnetic energy inside a length ℓ of the core via:

$$U = \frac{1}{2} \int \mathbf{J} \cdot \mathbf{A}\, dV$$

   Did you get the same result?

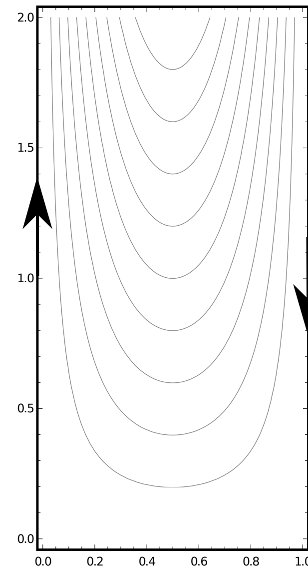
   (f) Finally using one the the above results calculate the self inductance (L) in a length ℓ.

2. Flux is Fun!

(a) Flux by Integration: Consider a rectangular loop 1×2 in the $z = 0$, xy plane with diagonal corners at $(0, 0)$ and $(1, 2)$. The magnetic field inside this loop in the $z = 0$ plane as a function of position and time is given by:

$$\mathbf{B} = B_0\, y \sin(\pi x) \left(\cos(\omega t)\, \widehat{\mathbf{j}} + \sin(\omega t)\, \widehat{\mathbf{k}}\right)$$

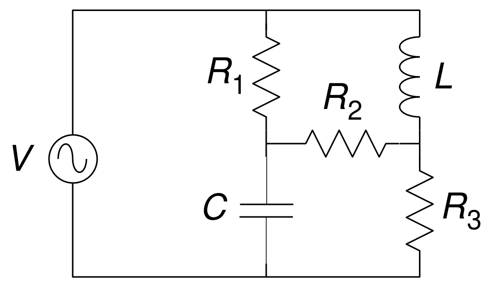
(So inside the rectangle the magnetic field is rotating in the yz plane, is zero on three of the edges and the large values of B are found near $(\frac{1}{2}, 2)$. I've attached a contour plot of B inside the rectangle which might help you visualize the function, but this has no consequence for the actual integral calculation.) The loop has clockwise orientation around the rectangle (so which way is the normal?). Integrate to find the magnetic flux.



(b) At $t = 0$ will the induced current be clockwise or counter-clockwise? Explain! Note that at $t = 0$ the flux is itself zero. Does this matter? Explain!

(c) (**Extra Credit**) You have quite naturally calculated the magnetic flux using the surface that is $z = 0$ inside the rectangle (if for no other reason because that's the only place I told you the value of $\mathbf{B}$). For an actual magnetic flux any surface that has the same boundary (the rectangle in this case) would produce the same flux. Prove this statement. (I can think of two proofs of this statement: one based on $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ and one based on $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, but any correct proof you come up with will be fine with me.)

3. Write down the complex node equations for the following circuit (one equation for each node, i.e., one equation for each unknown). Additionally draw one loop and report the resulting Kirchhoff Loop Law equation. Carefully define any symbols you use, perhaps by annotating on the below the circuit diagram (or re-draw the diagram on your answer sheet and annotate that). Note: the far left device in the circuit () is an ac voltage generator of the form $V = V_0 \cos(\omega t)$ where ω is the given angular frequency. (None of these equations need to be 'solved' just written down.)



Please sign the followign statment:

In answering these questions I have used no aids other then the textbook and my class notes.