Lorentz Transformation of a Tensor

Consider the antisymmetric matrix (tensor) F with components $F_{\mu\nu}$ given by:

$$F_{\mu\nu} = \begin{bmatrix} 0 & b_z & -b_y & -ie_x \\ -b_z & 0 & b_x & -ie_y \\ b_y & -b_x & 0 & -ie_z \\ ie_x & ie_y & ie_z & 0 \end{bmatrix}$$

```
In[1]:= fuv={{0,b3,-b2,-I e1},{-b3,0,b1,-I e2},{b2,-b1,0,-I e3},{I e1,I e2,I e3,0}}
```

Recall that in a "rotated" coordinate system, tensors transform via a similarity transform:

$$M' = O \cdot M \cdot O^{-1} = O \cdot M \cdot O^{\mathrm{T}}$$

Thus you read off $\mathbf{b}'$ and $\mathbf{e}'$ ($\mathbf{b}$ and $\mathbf{e}$ 3-vectors in the S' frame) using *Mathematica*:

`In[2]:= <<LorentzBoost.m` . . . Learn the function `boost[vx,vy,vz]`

`In[3]:= fpuv=boost[v,0,0].fuv.Transpose[boost[v,0,0]]` . . . `fpuv` is the matrix for $F_{\mu\nu}$ in the S' frame. Note that `fpuv[[1,2]]` is b'_z.

1. From the *Mathematica* output, write down each component of $\mathbf{b}'$ and $\mathbf{e}'$. Write down Eqs (22-40) from the textbook. Show that these two agree. (E.g., what is $\mathbf{u} \times \mathbf{B}$? What is $\mathbf{E}_\perp$? cs will need to be inserted as they are absent from `fuv`.)

Recall that we make invariant quantities by "using up" all the subscripted indexes (dot product), i.e.:

$$\beta \cdot \beta = \sum_{\mu=1}^{4} \beta_\mu \beta_\mu$$

Thus we can find an invariant from a matrix by "using up" the two subscripted indexes— taking the trace:

$$\mathrm{trace}(M) \equiv \sum_{\mu=1}^{4} M_{\mu\mu}$$

The trace of $F_{\mu\nu}$ (`Tr(fuv)` in Mathematica) is invariant but dull: 0.

2. What is the trace of $F \cdot F$? What is the trace of $F' \cdot F'$? Is the trace invariant?

The proof that trace produces invariants uses the following multiplicative property of trace:

$$\mathrm{trace}(A \cdot B) = \mathrm{trace}(B \cdot A)$$

Thus:

$$\mathrm{trace}(M') = \mathrm{trace}(O \cdot M \cdot O^{\mathrm{T}}) = \mathrm{trace}(O^{\mathrm{T}} \cdot O \cdot M) = \mathrm{trace}(O^{-1} \cdot O \cdot M) = \mathrm{trace}(M)$$

There is a similar multiplicative property for the determinant:

$$\det(A \cdot B) = \det(B \cdot A)$$

3. `Simplify` the `Det` of F and $F \cdot F$. `Simplify` the `Det` of F' and $F' \cdot F'$. What is the fundamental invariant?