Complex Numbers Review

Reference: Mary L. Boas, *Mathematical Methods in the Physical Sciences*
Chapter 2 & 14
George Arfken, *Mathematical Methods for Physicists*
Chapter 6

The real numbers (denoted $\mathbb{R}$) are incomplete in the sense that standard operations applied to some real numbers do not yield a real result (e.g., square root: $\sqrt{-1}$). It is surprisingly easy to enlarge the set of real numbers producing a set of numbers that *is* closed under standard operations: one simply needs to include $\sqrt{-1}$ (and linear combinations of it). Thus this enlarged field of numbers, called the *complex numbers* (denoted $\mathbb{C}$), consists of numbers of the form: $z = a + b\sqrt{-1}$ where a and b are real numbers. There are lots of notations for theses numbers. In mathematics, $\sqrt{-1}$ is called i (so $z = a + bi$), whereas in electrical engineering i is frequently used for current, so $\sqrt{-1}$ is called j (so $z = a + bj$). In *Mathematica* complex numbers are constructed using `I` for i. Since complex numbers require two real numbers to specify them they can also be represented as an ordered pair: $z = (a, b)$. In any case a is called the real part of z: $a = \mathrm{Re}(z)$ and b is called the imaginary part of z: $b = \mathrm{Im}(z)$. Note that the imaginary part of any complex number is real and the imaginary part of any real number is zero. Finally there is a polar notation which reports the radius (a.k.a. absolute value or magnitude) and angle (a.k.a. phase or argument) of the complex number in the form: $r\angle\theta$. The polar notation can be converted to an algebraic expression because of a surprising relationship between the exponential function and the trigonometric functions:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Thus there is a simple formula for the complex number z_1 in terms of its magnitude and angle:

$$\begin{aligned}
|z_1| &\equiv \sqrt{a^2+b^2} = r \\
a &= r\cos\theta = |z_1|\cos\theta \\
b &= r\sin\theta = |z_1|\sin\theta \\
z_1 &= a + bj = |z_1|(\cos\theta + j\sin\theta) = |z_1|e^{j\theta}
\end{aligned}$$

For example, we have the following notations for the complex number $1 + i$:

$$1 + i = 1 + j = 1 + \mathtt{I} = (1, 1) = \sqrt{2}\angle 45^\circ = \sqrt{2}e^{j\pi/4}$$

Since complex numbers are closed under the standard operations, we can define things which previously made no sense: $\log(-1)$, $\arccos(2)$, $(-1)^\pi$, $\sin(i)$, $\ldots$. The complex numbers are large enough to define every function value you might want. Note that addition, subtraction, multiplication, and division of complex numbers proceeds as usual, just using the symbol for $\sqrt{-1}$ (let's use j):

$$z_1 = a + bj \qquad z_2 = c + dj$$

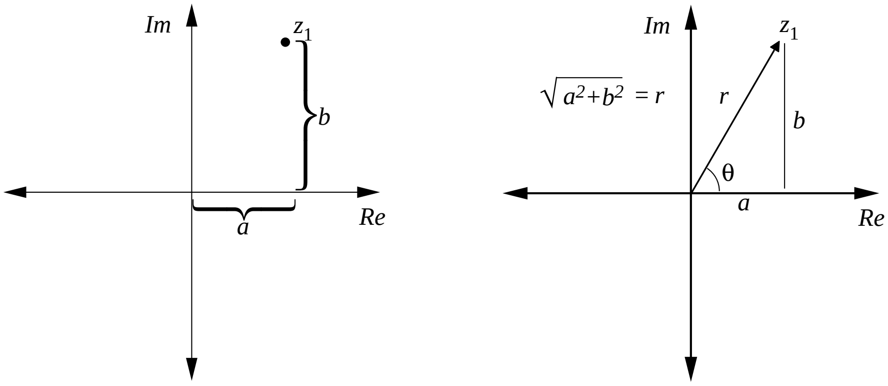


Figure 1: Complex numbers can be displayed on the complex plane. A complex number $z = a + bi$ may be displayed as an ordered pair: (a, b), with the "real axis" the usual x-axis and the "imaginary axis" the usual y-axis. Complex numbers are also often displayed as vectors pointing from the origin to (a, b). The angle θ can be found from the usual trigonometric functions; $|z| = r$ is the length of the vector.

$$
\begin{aligned}
z_1 + z_2 &= (a + bj) + (c + dj) = (a + c) + (b + d)j \\
z_1 - z_2 &= (a + bj) - (c + dj) = (a - c) + (b - d)j \\
z_1 \times z_2 &= (a + bj) \times (c + dj) = ac + adj + bcj + bdj^2 = (ac - bd) + (ad + bc)j \\
\frac{1}{z_1} &= \frac{1}{a + bj} = \frac{1}{a + bj} \times \frac{a - bj}{a - bj} = \frac{a - bj}{a^2 + b^2} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}\, j
\end{aligned}
$$

Note in calculating $1/z_1$ we made use of the complex number $a - bj$; $a - bj$ is called the *complex conjugate* of z_1 and it is denoted by z_1^* or sometimes $\overline{z_1}$. See that $zz^* = |z|^2$. Note that, in terms of the ordered pair representation of $\mathbb{C}$, complex number addition and subtraction looks just like component-by-component vector addition:

$$(a, b) + (c, d) = (a + b, c + d)$$

Thus there is a tendency to denote complex numbers as vectors rather than points in the complex plane.

While the closure property of the complex numbers is dear to the hearts of mathematicians, the main use of complex numbers in science is to represent sinusoidally varying quantities in a simple way. For example, you may remember that the superposition of sinusoidal quantities is itself sinusoidal, but with a new amplitude and phase. For example, in a series RC circuit the voltage across the resistor might be given by $A \cos \omega t$ whereas the voltage across the capacitor might be given by $B \sin \omega t$, and the voltage across the combination (according to Kirchhoff) is the sum:

$$
\begin{aligned}
V_R(t) + V_C(t) &= A \cos \omega t + B \sin \omega t \qquad \text{where: } A, B \in \mathbb{R} \\
&= \sqrt{A^2 + B^2} \left(\frac{A}{\sqrt{A^2 + B^2}} \cos \omega t + \frac{B}{\sqrt{A^2 + B^2}} \sin \omega t \right) \\
&= \sqrt{A^2 + B^2} \left(\cos \delta \cos \omega t + \sin \delta \sin \omega t \right) \qquad \text{where: } \cos \delta = \frac{A}{\sqrt{A^2 + B^2}} \\
&= \sqrt{A^2 + B^2} \cos(\omega t - \delta)
\end{aligned}
$$

Yuck! That's a lot of work just to add two sinusoidal waves; we seek a simpler method (which might not seem overly simple at first glance). Note that V_R can be written as

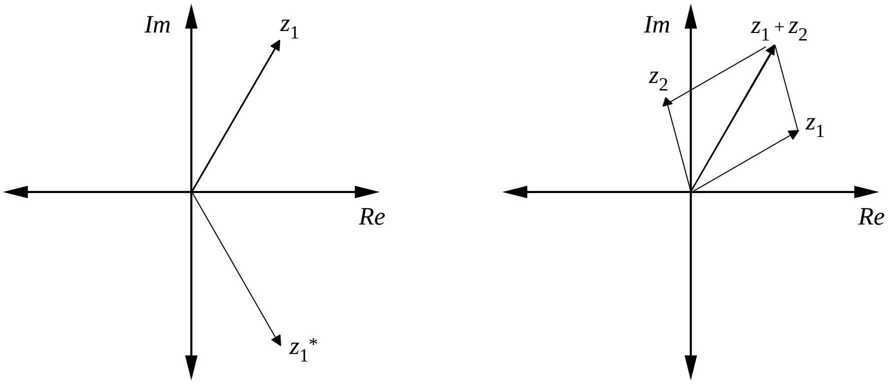


Figure 2: The complex conjugate is obtained by reflecting the vector in the real axis. Complex number addition works just like vector addition.

$\mathrm{Re}(Ae^{j\omega t})$ and V_C can be written as $\mathrm{Re}(-jBe^{j\omega t})$ so:

$$V_R(t) + V_C(t) = \mathrm{Re}\left((A - jB)e^{j\omega t}\right)$$

Now using the polar form of the complex number $A - jB$:

$$A - jB = \sqrt{A^2 + B^2}\, e^{-j\delta} \qquad \text{where: } \tan\delta = B/A$$

we have:

$$\begin{aligned} V_R(t) + V_C(t) &= \mathrm{Re}\left((A - jB)e^{j\omega t}\right) \\ &= \mathrm{Re}\left(\sqrt{A^2 + B^2}\, e^{-j\delta}\, e^{j\omega t}\right) \\ &= \sqrt{A^2 + B^2}\, \mathrm{Re}\left(e^{j(\omega t - \delta)}\right) \\ &= \sqrt{A^2 + B^2}\, \cos(\omega t - \delta) \end{aligned}$$

Complex numbers are particularly important for calculations in a.c. circuits, where voltages and currents are all changing sinusoidally at the same frequency. We assume each is of the form:

$$\begin{aligned} v(t) &= \mathrm{Re}\left(V_0\, e^{j\omega t}\right) \\ i(t) &= \mathrm{Re}\left(I_0\, e^{j\omega t}\right) \end{aligned}$$

The possibility of phase shifts between these voltages and currents is accounted for by making V_0 and I_0 complex numbers:

$$\begin{aligned} v(t) &= \mathrm{Re}\left(V_0 e^{j\omega t}\right) \\ &= \mathrm{Re}\left(|V_0| e^{j\phi} e^{j\omega t}\right) \\ &= |V_0| \cos(\omega t + \phi) \end{aligned}$$

Thus ϕ would be the phase shift of this voltage and $V_{rms} = |V_0|/\sqrt{2}$.

In the case of a capacitor, the voltage depends on the stored charge, which is the integral of the current:

$$v(t) = \frac{q(t)}{C} = \frac{1}{C}\int i\,dt = \frac{1}{C}\,\mathrm{Re}\left(\int I_0 e^{j\omega t}\,dt\right) = \mathrm{Re}\left(\frac{I_0}{j\omega C}\,e^{j\omega t}\right)$$

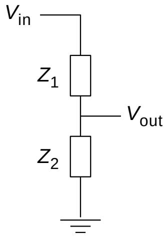
So $V_0 = I_0/j\omega C$, i.e., voltage and current have a linear relationship. Playing the role of resistance is $Z = 1/(j\omega C)$, which is called the impedance of the capacitor. For resistors, capacitors and inductors there is a linear relationship between the complex currents flowing through the device and the complex voltage across the device:

$$V_0 = ZI_0$$

where Z is the complex impedance. For resistors $Z = R$, for capacitors $Z = 1/(j\omega C)$ and for inductors $Z = j\omega L$.

The complex numbers V_0, I_0, and Z can be treated in Kirchhoff's laws exactly as voltages, currents, and resistances were treated in d.c. circuits. Thus for a general voltage divider we have:

$$\begin{aligned} V_{\text{out}} &= Z_2 I = Z_2\,\frac{V_{\text{in}}}{Z_1+Z_2} \\ \frac{V_{\text{out}}}{V_{\text{in}}} &= \frac{Z_2}{Z_1+Z_2} \end{aligned}$$



So if Z_2 is a capacitor and Z_1 is a resistor (i.e., our low pass filter) we have:

$$\begin{aligned} \frac{V_{\text{out}}}{V_{\text{in}}} &= \frac{1/(j\omega C)}{R+1/(j\omega C)} \\ &= \frac{1}{j\omega RC+1} \\ &= \frac{1}{|j\omega RC+1|e^{j\delta}} \qquad \text{where: } \tan\delta = \omega RC \\ &= \frac{1}{\sqrt{(\omega RC)^2+1}}\,e^{-j\delta} \end{aligned}$$

See that the -3 dB frequency (where $|V_{\text{out}}/V_{\text{in}}| = 1/\sqrt{2}$) must satisfy: $\omega RC = 1$. If $\omega \ll 1/RC$ (i.e., low frequency) we have:

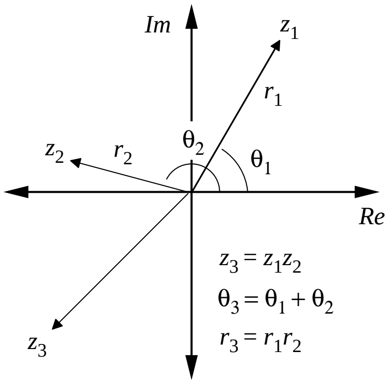
$$\frac{V_{\text{out}}}{V_{\text{in}}} \approx 1$$

If $\omega \gg 1/RC$ (high frequency) we have:

$$\frac{V_{\text{out}}}{V_{\text{in}}} \approx \frac{1}{j\omega RC}$$

Homework

1. Prove that when you multiply complex numbers z_1 and z_2, the magnitude of the result is the product of the magnitudes of z_1 and z_2, and the phase of the product is the sum of the phases of z_1 and z_2.



2. Express the following in the $r\angle\theta$ format (I bet your calculator can do this automatically):
   (a) $\dfrac{1}{1+i}$ (b) $\dfrac{3+i}{1+3i}$ (c) $25e^{2i}$ (d) $(1/(1+i))^*$ (e) $\left|\dfrac{1}{(1+i)}\right|$

3. Find the following in (a, b) format (I bet your calculator can do this automatically):
   (a) $\dfrac{3i-7}{i+4}$ (b) $(.64+.77i)^4$ (c) $\sqrt{3+4i}$ (d) $25e^{2i}$ (e) $\log(-1)$

4. Consider the following circuit. Plot the $(V_{\text{out}})_{rms}$ as a function of frequency, where $(V_{\text{in}})_{rms} = 1$ V. Plot the phase difference between V_{out} and V_{in} as a function of frequency. Your plotted frequency range should include frequencies such that $X_C \gg X_L$ and $X_C \ll X_L$.