Class 9: hamiltonian70.pdf #: 1, 2, 3; old exam #1, 13.23

1) $T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2}\frac{I}{r^2}\dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2$

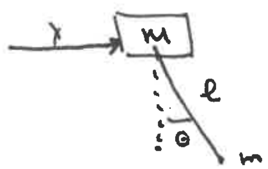
$U = \frac{1}{2} k x^2 - m_2 g x$

$$L = \frac{1}{2}\underbrace{\left(m_1 + m_2 + I/r^2\right)}_{\text{call this } m}\dot{x}^2 + m_2 g x - \frac{1}{2} k x^2$$

$p_x = m\dot{x}$

$H = p_x \dot{x} - L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 - m_2 g x$ now express in terms of p_x

$= \frac{1}{2}\frac{p_x^2}{m} + \frac{1}{2} k x^2 - m_2 g x$

2) 

Locate M: $X \to T = \frac{1}{2} M \dot{X}^2$

Locate m: $x = X + l\sin\theta \to \dot{x} = \dot{X} + l\sin\theta\,\dot{\theta}$

$y = -l\cos\theta \to \dot{y} = l\sin\theta\,\dot{\theta}$

$\dot{x}^2 + \dot{y}^2 = \dot{X}^2 + (l\dot{\theta})^2 + 2l\sin\theta\,\dot{X}\dot{\theta}$

total $T = \frac{1}{2} M\dot{X}^2 + \frac{1}{2} m\left(\dot{X}^2 + (l\dot{\theta})^2 + 2l\sin\theta\,\dot{X}\dot{\theta}\right)$

$U = -mgl\cos\theta$

$$L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2} m\left((l\dot{\theta})^2 + 2l\sin\theta\,\dot{x}\dot{\theta}\right) + mgl\cos\theta$$

$p_x = (M+m)\dot{x} + ml\sin\theta\,\dot{\theta}$ $p_\theta = ml^2\dot{\theta} + ml\sin\theta\,\dot{x}$

$p_x\dot{x} + p_\theta\dot{\theta} = (M+m)\dot{x}^2 + 2ml\sin\theta\,\dot{x}\dot{\theta} + ml^2\dot{\theta}^2$

$p_x\dot{x} + p_\theta\dot{\theta} - L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2} m\left((l\dot{\theta})^2 + 2l\sin\theta\,\dot{x}\dot{\theta}\right) - mgl\cos\theta$

needs to be in terms of p_x & p_θ

$$\left.\begin{aligned} p_x &= (M+m)\dot{x} + ml\sin\theta\,\dot{\theta} \\ p_\theta &= ml^2\dot{\theta} + ml\sin\theta\,\dot{x} \\ &= ml\sin\theta\,\dot{x} + ml^2\dot{\theta} \end{aligned}\right]$$

2 eqns
2 unknowns

$$\dot{x} = \frac{\begin{vmatrix} p_x & ml\sin\theta \\ p_\theta & ml^2 \end{vmatrix}}{\begin{vmatrix} M+m & ml\sin\theta \\ ml\sin\theta & ml^2 \end{vmatrix}} = \frac{p_x ml^2 - p_\theta ml\sin\theta}{(M+m)ml^2 - m^2l^2\sin^2\theta} = \frac{p_x - p_\theta/l\,\sin\theta}{M + m\cos^2\theta}$$

$$\dot{\theta} = \frac{\begin{vmatrix} (M+m) & p_x \\ ml\sin\theta & p_\theta \end{vmatrix}}{\begin{vmatrix} M+m & ml\sin\theta \\ ml\sin\theta & ml^2 \end{vmatrix}} = \frac{(M+m)p_\theta - ml\sin\theta\,p_x}{(M+m)ml^2 - m^2l^2\sin^2\theta}$$

$(M+m)ml^2 - m^2l^2\sin^2\theta = ml^2(M + m\cos^2\theta)$

$$P_x \dot{x} + P_\theta \dot{\theta} = \frac{P_x\left(P_x - \frac{P_\theta}{\ell}\sin\theta\right) + P_\theta\left((M+m)P_\theta - m\ell\sin\theta P_x\right)/m\ell^2}{M + m\cos^2\theta}$$

$$= \frac{P_x^2 - 2\frac{P_x P_\theta}{\ell}\sin\theta + P_\theta^2\frac{(M+m)}{m\ell^2}}{M + m\cos^2\theta} \quad \leftarrow \text{this is } 2T$$

$$H = \frac{P_x^2 - 2\frac{P_x P_\theta}{\ell}\sin\theta + P_\theta^2\frac{(M+m)}{m\ell^2}}{2(M + m\cos^2\theta)} + mg\ell\cos\theta$$

↑ ugly

3) $\vec{v} = \dot{r}\hat{r} + \dot{z}\hat{z} + (r\dot{\phi})\hat{\phi} \qquad z = f(r) \rightarrow \dot{z} = f'\dot{r}$

$$v^2 = \dot{r}^2 + \dot{z}^2 + (r\dot{\phi})^2 = \dot{r}^2(1+f'^2) + (r\dot{\phi})^2$$

$$L = \frac{1}{2}m\left(\dot{r}^2(1+f'^2) + (r\dot{\phi})^2\right) - mgf(r)$$

$$P_r = m(1+f'^2)\dot{r} \qquad P_\phi = mr^2\dot{\phi}$$

$$H = \frac{P_r^2}{2m(1+f'^2)} + \frac{P_\phi^2}{2mr^2} + mgf(r)$$

since a standard time independent coordinate transform. $H = E = \text{const}$

since ϕ is cyclic $\rightarrow P_\phi = \text{const}$

so: $E = \frac{1}{2}m(1+f'^2)\dot{r}^2 + \frac{P_\phi^2}{2mr^2} + mgf(r)$

$$\frac{E - \frac{P_\phi^2}{2mr^2} - mgf(r)}{\frac{1}{2}m(1+f'^2)} = \dot{r}^2$$

$$dt = \frac{dr}{\sqrt{\frac{E - \frac{P_\phi^2}{2mr^2} - mgf(r)}{\frac{1}{2}m(1+f'^2)}}} \quad \leftarrow \text{integrate me}$$

old exam #1

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \underbrace{Bx\dot{x}}_{} - V(x,y)$$

unneeded remark: this is $\frac{d}{dt}\left(\frac{1}{2}x^2\right)$

so Action $\int \frac{d}{dt}\left(\frac{1}{2}x^2\right)dt = \frac{1}{2}x^2\Big|_i^f$

indep of δx

so has no effect on motion!

$$\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} + B\dot{x} = \frac{d}{dt}\left(m\dot{x} + Bx\right) = m\ddot{x} + B\dot{x}$$

cancels

$$\frac{\partial L}{\partial y} = -\frac{\partial V}{\partial y} = \frac{d}{dt}(m\dot{y}) = m\ddot{y}$$

SO: $m\vec{a} = -\vec{\nabla}V$ with no B effects

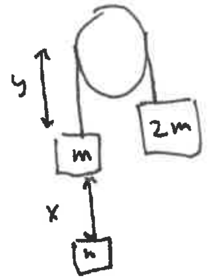
$$P_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + Bx \qquad P_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

$$H = P_x\dot{x} + P_y\dot{y} - L = (m\dot{x}+Bx)\dot{x} + m\dot{y}\dot{y} - \left[\frac{1}{2}m(\dot{x}^2+\dot{y}^2) + Bx\dot{x} - V\right]$$

$= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + V$ ← looks just like energy

$= \dfrac{(P_x - Bx)^2}{2m} + \dfrac{P_y^2}{2m} + V$ ← now doesn't look like energy but still is!

13.23



location bottom m: $-y-x+\text{const}$

top m: $-y$

2m: $+y+\text{const}$

gravity PE $= mg(-y-x) + gm(-y) + 2mg(y)$

$= -mgx + \text{const}$

Spring: $\frac{1}{2}kx^2$

$KE = \frac{1}{2}m(\dot{y}+\dot{x})^2 + \frac{1}{2}m\dot{y}^2 + m\dot{y}^2$

$$L = 2m\dot{y}^2 + \frac{1}{2}m\dot{x}^2 + m\dot{x}\dot{y} - \frac{1}{2}kx^2 + mgx$$

y cyclic $\Rightarrow \frac{\partial L}{\partial \dot{y}} = P_y = \text{Const} = 4m\dot{y} + m\dot{x}$

$\frac{\partial L}{\partial \dot{x}} = P_x = m\dot{x} + m\dot{y}$

$P_y = m\dot{x} + 4m\dot{y}$

$P_x = m\dot{x} + m\dot{y}$

$P_y - P_x = 3m\dot{y}$

$4P_x - P_y = 3m\dot{x}$

$$2T = P_x\dot{x} + P_y\dot{y} = P_x\left(\frac{4P_x - P_y}{3m}\right) + P_y\left(\frac{P_y - P_x}{3m}\right)$$

$$= \frac{4P_x^2 - 2P_xP_y + P_y^2}{3m}$$

$$H = T + U = \frac{4P_x^2 - 2P_xP_y + P_y^2}{6m} + \frac{1}{2}kx - mgx$$

$-\frac{\partial H}{\partial x} = \dot{P}_x = -kx + mg$

$-\frac{\partial H}{\partial y} = \dot{P}_y = 0 \rightarrow P_y = \text{constant}$

$\frac{\partial H}{\partial P_x} = \dot{x} = \frac{4}{3m}P_x - \frac{P_y}{3m}$ ← same as above

$\frac{\partial H}{\partial y} = \dot{P} = \frac{-P_x}{3m} + \frac{P_y}{3m}$ ← same as above

$P_y = 0$ always

$P_x(t=0) = 0$

I need initial conditions: $y=0$, $\dot{y}=0$, $x=?$, $\dot{x}=0$

↳ equilibrium would have $x = \frac{mg}{k}$

apparently we have x_0 more

$x(0) = \frac{mg}{k} + x_0$

$P_y = 0$ always $\Rightarrow \frac{3m}{4}\dot{x} = P_x$

$\dot{P}_x = \frac{3m}{4}\ddot{x} = -kx + mg$ ← particular solution $x = \frac{mg}{k}$ const

homo solution: $\ddot{x} = -\underbrace{\frac{4k}{3m}}_{\text{define as } \omega^2}x$ ← $x = A\cos(\omega t + \delta)$ ← if $\dot{x}(0) = 0$ $\delta = 0$

General solution: $x = \frac{mg}{k} + A\cos(\omega t + \delta) \rightarrow A = x_0,\ \delta = 0$

$P_y = 0 \Rightarrow y = -\frac{1}{4}x + \text{const} = -\frac{1}{4}(x - x_0)$