Class 4: old exam #3, 4.2, 4.8, 4.12, 4.23

4.2 – to do a line integral we need to parametrize the path

[express $\vec{r} = \vec{r}(t)$] $\qquad \int \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt$

← not required to do this

(a)  express the path in 2 pieces $\quad \vec{r}_1 = (t, 0)$ & $\vec{r}_2 = (1, t)$

$\frac{d\vec{r}_1}{dt} = (1,0) \qquad \frac{d\vec{r}_2}{dt} = (0,1)$

↑ here $x = t$, $y = 0$ $\qquad$ here $x = 1$, $y = t$

$$\int_0^1 \vec{F} \cdot \frac{d\vec{r}_1}{dt}\, dt + \int_0^1 \vec{F} \cdot \frac{d\vec{r}_2}{dt}\, dt$$

$$\int_0^1 F_x(\cancel{t}, 0)\, dt + \int_0^1 F_y(1, t)\, dt$$

↑ $x = t$

$$\int_0^1 t^2\, dt + \int_0^1 2t\, dt = \frac{1}{3} + 1 = 4/3$$

(b) $\vec{r} = (t, t^2)$ $\qquad \int_0^1 \vec{F}(t, t^2) \cdot (1, 2t)\, dt$

$\frac{d\vec{r}}{dt} = (1, 2t)$ $\qquad \int_0^1 (t^2 + 2t^3 \cdot t)\, dt = \frac{1}{3} + \frac{2}{5} = \frac{11}{15}$

$x = t$
$y = t^2$
so $y = x^2$

(c) $\vec{r} = (t^3, t^2)$ $\qquad \int_0^1 F(t^3, t^2) \cdot (3t^2, 2t)\, dt$

$\frac{d\vec{r}}{dt} = (3t^2, 2t)$ $\qquad \int_0^1 (t^6 \cdot 3t^2 + 2t^5 \cdot 2t)\, dt$

$$\frac{3}{9} + \frac{4}{7} = \frac{1}{3} + \frac{4}{7} = \frac{7+12}{21} = \frac{19}{21}$$

Note: $\nabla \times F \neq 0$ so <u>NOT</u> path indep

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^2 & 2xy & 0 \end{vmatrix} = \hat{k}\, 2y$$

4.12: $\vec{\nabla}(x^2+z^3) = (2x, 0, 3z^2)$

$\vec{\nabla}\, ky = (0, k, 0)$

$\vec{\nabla} \sqrt{x^2+y^2+z^2} = \left(\frac{x}{\sqrt{}}, \frac{y}{\sqrt{}}, \frac{z}{\sqrt{}}\right) = \frac{\vec{r}}{r} = \hat{r}$

$\vec{\nabla} \frac{1}{\sqrt{}} = -\left(\frac{x}{\sqrt{}^3}, \frac{y}{\sqrt{}^3}, \frac{z}{\sqrt{}^3}\right) = \frac{-\vec{r}}{r^3}$

4.23 check that $\vec{\nabla}\times\vec{F} = 0$ $\quad u = -k\left(\frac{x^2}{2}+y^2+\frac{3}{2}z^2\right)$

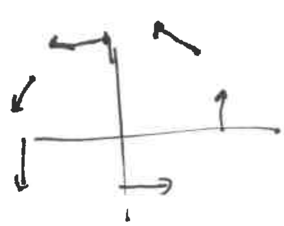
(a) $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \cancel{kx} & \cancel{2ky} & \cancel{3kz} \\ kx & 2ky & 3kz \end{vmatrix} = \hat{i}\cdot 0 + \hat{j}\cdot 0 + \hat{k}\cdot 0$ ← consv.

(b) $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cancel{ky} & \cancel{k} & \\ \partial_x & \partial_y & \partial_z \\ ky & kx & 0 \end{vmatrix} = \hat{i}\cdot 0 + \hat{j}\cdot 0 + \hat{k}\,(k-k) = 0$ ← cons

↖ eg $u = -kxy$

(c) look at a display of this F

It clearly has "curl"



$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -ky & kx & 0 \end{vmatrix} = \hat{i}\cdot 0 + \hat{j}\cdot 0 + \hat{k}\,2k$

↑ not zero

Not consv

Note: we did this in class 4

Physics 339 Problem 4.8 September 20014

Please note the connection between problem 4.8 and the material we've already covered. Eq. (1.47) relates acceleration in polar coordinates:

$$\ddot{\mathbf{r}} = \mathbf{a} = (\ddot{r} - r\dot{\phi}^2)\,\hat{\mathbf{r}} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\,\hat{\boldsymbol{\phi}}$$

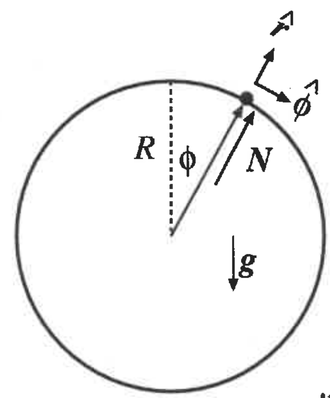
In the context of this problem r is the constant R (at least while still in contact with the sphere) so:

$$\ddot{\mathbf{r}} = \mathbf{a} = (-R\dot{\phi}^2)\,\hat{\mathbf{r}} + (R\ddot{\phi})\,\hat{\boldsymbol{\phi}}$$

and while a sphere is named in the problem, the motion will be 'straight down' i.e., on a circle (so we can use polar coordinates).

Example 1.2 describes a skateboard oscillating around the bottom of a pipe. This is essentially the opposite of our problem. Example 1.2 defines ϕ from the bottom of the pipe; in problem 4.8 you'll want to define ϕ from the top of the sphere.

The location of the particle on the sphere is defined by ϕ:



$$\begin{aligned}
\mathbf{r} &= R\sin\phi\,\hat{\mathbf{i}} + R\cos\phi\,\hat{\mathbf{k}} = R\,\hat{\mathbf{r}} \\
\mathbf{v} &= R\cos\phi\,\dot{\phi}\,\hat{\mathbf{i}} - R\sin\phi\,\dot{\phi}\,\hat{\mathbf{k}} = R\dot{\phi}\,\hat{\boldsymbol{\phi}} \\
\mathbf{a} &= (-R\dot{\phi}^2)\,\hat{\mathbf{r}} + (R\ddot{\phi})\,\hat{\boldsymbol{\phi}}
\end{aligned}$$

$v^2 = (R\dot{\phi})^2$

The total force on the particle is:

$$\begin{aligned}
\mathbf{F} &= m\mathbf{g} + \mathbf{N} \\
&= mg(-\cos\phi\,\hat{\mathbf{r}} + \sin\phi\,\hat{\boldsymbol{\phi}}) + N\hat{\mathbf{r}} = m\ddot{\vec{r}} = m(-R\dot{\phi}^2\hat{r} + R\ddot{\phi}\hat{\phi})
\end{aligned}$$

Look at $\hat{r}$: $-mg\cos\phi + N = m(-R\dot{\phi}^2)$

If $N > 0$ we have the usual situation of the sphere pushing the particle out from the surface. If $N < 0$ we have the impossible situation of the sphere sucking the particle into the surface. Evidently the moment when $N = 0$ is the moment that the particle leaves the surface of the sphere. $-mg\cos\phi + N = -mR\left(\frac{g}{R} - \frac{g}{R}\cos\phi\right)2$

As stated in the problem, by using conservation of energy you should be able to calculate v for any angle ϕ, and from that calculate $\dot{\phi}$ for any angle ϕ.

divide by mg : $\frac{N}{mg} = 3\cos\phi - 2$

By looking at the radial component for the equation $\mathbf{F} = m\mathbf{a}$ you should then be able to find the angle at which $N = 0$.

Conservation of energy ⇒ $PE = mgh = mgR\cos\phi$ initial PE

$KE = \frac{1}{2}mv^2 = \frac{1}{2}mR^2\dot{\phi}^2$

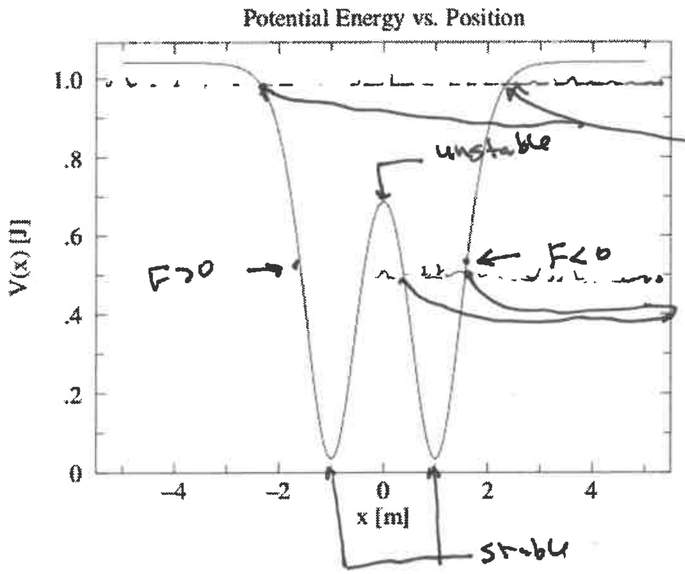
So $\cos\phi = \frac{2}{3}$ is where $N = 0$

$PE + KE = mgR\cos\phi + \frac{1}{2}mR^2\dot{\phi}^2 = mgR$

or $\frac{g}{R}\cos\phi + \frac{1}{2}\dot{\phi}^2 = \frac{g}{R}$

3. The following plots display the potential energy (in J) of a particular force as a function of x measured in meters. The second plot displays a detail near $x = 1$ of the first.

1.5 – unbounded motion – lots of KE goes either $+\infty$ or $-\infty$



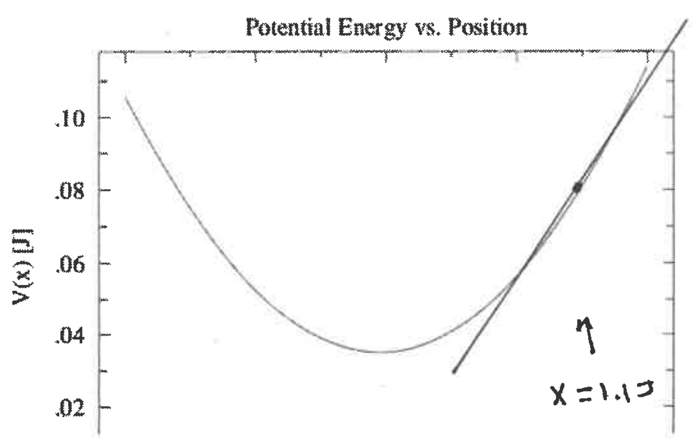
turning points – bounces between going slow @ $x=0$ & fast @ $x = \pm 1$

bounces back/forth between turning pts in rhs well

(a) Report: an x value that is a stable equilibrium point, an x value that is an unstable equilibrium point, an x value for which the force pushes in the positive x direction, and an x value for which the force pushes in the negative x direction. The potential energy plot is quite flat for $|x| > 5$, but remains at a value a bit above 1 J. What can you conclude about the force in the region $|x| > 5$?

if $u = \text{const}$ $-\frac{du}{dx} = F = 0$

(b) Describe the future trajectory of a particle released at $x = 1$ with a total energy of 0.5 J. Describe the future trajectory of a particle released at $x = 1$ with a total energy of 1.0 J. Describe the future trajectory of a particle released at $x = 1$ with a total energy of 1.5 J.



Seek slope @ $x = 1.15$

$u(1.2) \sim .11$

$u(1.1) \sim .06$

$\frac{\Delta u}{\Delta x} = \frac{.05}{.1} = .5 \frac{J}{m} = .5\,N$

Note since $F = -\frac{du}{dx}$ this force is negative i.e. points to left