Class 11: 10.35, 37 old exam: 1, 2, tetrahedron, matrix problems

10.35

m	$\vec{r}$
m	$(a, 0, 0)$
$2m$	$(0, a, a)$
$3m$	$(0, a, -a)$

$$\overleftrightarrow{I} = \sum m_\alpha \left(r_\alpha^2 \overleftrightarrow{1} - \vec{r}_\alpha \vec{r}_\alpha \right)$$

$$I = m \begin{pmatrix} 0 & & \\ & a^2 & \\ & & a^2 \end{pmatrix} + 2m \begin{pmatrix} 2a^2 & 0 & 0 \\ 0 & a^2 & -a^2 \\ 0 & -a^2 & a^2 \end{pmatrix} + 3m \begin{pmatrix} 2a^2 & 0 & 0 \\ 0 & a^2 & a^2 \\ 0 & a^2 & a^2 \end{pmatrix}$$

$$= ma^2 \begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix}$$

easy eigenvector: $10ma^2 \longleftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

easy guess: $7ma^2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

last must be $\perp$ to both: $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \longleftrightarrow 5ma^2$

without guessing: $\begin{pmatrix} 10-x & & \\ & 6-x & 1 \\ & 1 & 6-x \end{pmatrix} = (10-x)(6-x)^2 - 10 - x$

$= (10-x)\left[(6-x)^2 - 1\right]$

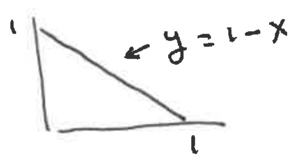
$x^2 - 12x + 35 = (x-7)(x-5)$

to solve for an eigenvector, plug in x & seek nontrivial soln to homo eq

eg $x = 5$: $\begin{pmatrix} 5 & & \\ & 1 & 1 \\ & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$ $\quad 5a = 0 \rightarrow a = 0$

$b + c = 0 \Rightarrow b = -c$

Note: overall scale of eigenvector is undetermined; often folks normalize (i.e., make eigenvector a unit vector) this is not required here.

10.37 

Area $= \frac{1}{2}$

$$I_{xx} = \int_0^1 \int_0^{1-x} dx\,dy\; x^2\; 24$$
$$= \int_0^1 (1-x)x^2\, dx\; 24 = 24\underbrace{\left(\frac{1}{3} - \frac{1}{4}\right)}_{\frac{1}{12}}$$
$$= 2$$

$$I_{zz} = I_{xx} + I_{yy} = 4$$

$$I_{yy} = \int_0^1 \int_0^{1-x} dx\,dy\; y^2\; 24$$
$$= \int_0^1 \frac{(1-x)^3}{3} \cdot 24\, dx = -\frac{(1-x)^4}{12} \cdot 24$$

$$-I_{xy} = \int_0^1 \int_0^{1-x} dx\,dy\; xy\; 24$$
$$= \int_0^1 \frac{(1-x)^2}{2} x\; 24 = \int_0^1 (x - 2x^2 + x^3)12 = 2$$
$$= \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)12 = 1$$
$$\frac{6-8+3}{12} = \frac{1}{12}$$

$$I = \begin{pmatrix} 2 & -1 & \\ -1 & 2 & \\ & & 4 \end{pmatrix}$$

moment	vector
4	$(0, 0, 1)$
1	$(1, 1, 0)$
3	$(1, -1, 0)$

↑ easy to guess

The tetrahedron is the simplest of the platonic solids: four equilateral triangles combined into a pyramid structure. It can also be thought of as a cube's top diagonal and a twisted bottom diagonal. There are 24 symmetry operations that leave a tetrahedron invariant (i.e., result in a shuffling of the vertices within in the same locations). Matrices are designed to perform such transformations. We examine below the symmetry operations that just involve rotations.

Find the spreadsheet `tetrahedron.xlsx` from the class web site.The upper lhs contains cells to hold the values for the three Euler Angles: ϕ, θ, ψ; columns F–H contain the 3×3 rotation matrix defined by those Euler Angles. (Note use `=pi()` in the spreadsheet not some approximation of π.) The (x, y, z) location of the four vertices of the tetrahedron (labeled: A, B, C, D) occur below the matrix. Matrix multiplication results in the transformed point (x', y', z'). (Check out the spreadsheet formulas or decode the self-documentation.) A symmetry operation on the tetrahedron's vertices should just shuffle those points (i.e., each output point (x', y', z') should be one of the A,B,C,D inputs). Enter each of the below Euler Angle triplets, find which vertex (A,B,C,D) corresponds to the (x', y', z') output and record the results in the below table. The Trace (Trace $\equiv$ sum of the diagonal elements) of a rotation matrix can be shown to be equal to $1 + 2\cos\alpha$ where α is the overall rotation angle of the matrix. Modify the spreadsheet to automatically calculate the Trace and calculate the corresponding angle α. Record these results also in the below table. Try to identify the single rotation axis that would do the transformation if a rotation by α were applied to that axis.

Find below the Eular Angles defining 11 rotation symmetry operations. Fill in the table showing where each vertex (A, B, C, D) is mapped by the operation, the Trace and rotation angle. I've filled in the first row as an example.

ϕ	θ	ψ	A	B	C	D	Tr	α	axis
π	0	0	B	A	D	C	-1	180° z	
0	π	0	C	D	A	B	-1	180 y	
$-\pi/2$	π	$\pi/2$	D	C	B	A	-1	180 x	
$\pi/2$	$\pi/2$	0	C	A	B	D	0	120 diagonal	D fixed
π	$\pi/2$	$\pi/2$	B	C	A	D	0	120 rems	
$-\pi/2$	$\pi/2$	π	B	B	C	A	0	120	C fixed
0	$\pi/2$	$-\pi/2$	D	A	C	B	0	120	
$-\pi/2$	$\pi/2$	0	D	B	A	C	0	120	B fixed
π	$\pi/2$	$-\pi/2$	C	B	D	A	0	120	
$\pi/2$	$\pi/2$	π	A	C	D	B	0	120	A fixed
0	$\pi/2$	$\pi/2$	A	D	B	C			

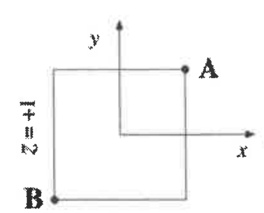
These transformations may make more sense to you if you think about rotating the *object* in the opposite order. You usually think of a tetrahedron sitting on its triangular base. Try: $\phi = \pi/3, \theta = \arctan(\sqrt{2}), \psi = -\pi/4$. **Q:** Which vertex is on top? Which bottom vertex is has $y' = 0$?

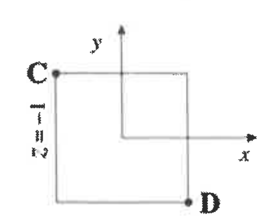
↑ D

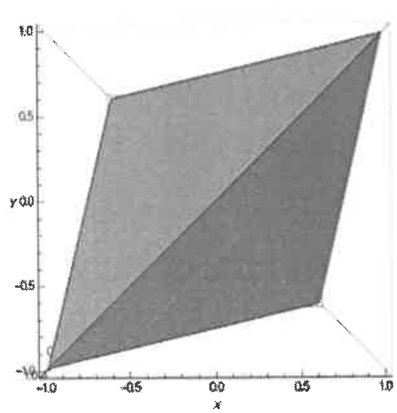
·A

C· ·D

$\theta = 56.6°$







matrix $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & 3 \end{pmatrix}$ $C = \begin{pmatrix} 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{pmatrix}$

Some important results from linear:

$\det(AB) = \det(A)\det(B) \longrightarrow \det(A^{-1}) = \frac{1}{\det A}$; $\det(A^T) = \det(A)$

$Tr(AB) = Tr(BA)$

$$\det\begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix} = 3 + 0 + 4 - 0 - 0 - -6 = 13$$

$$\det\begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & 3 \end{pmatrix} = -6 + 2 + 0 - 4 - 0 - 0 = -8$$

$$AB = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -2 & 1 \\ 1 & -2 & 9 \\ 5 & 3 & 3 \end{pmatrix}$$

$\leftarrow \det = -104 = -8 \times 13$

Trace = 0

$$A \cdot C \quad \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 7 \\ 13 & 9 \\ 5 & 2 \end{pmatrix}$$

$$C^T \cdot A \quad \begin{pmatrix} 2 & 4 & 1 \\ 1 & 3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 16 & 3 \\ 1 & 11 & 2 \end{pmatrix}$$

$$B^{-1} \overset{\text{mathematica}}{=} \frac{1}{8}\begin{pmatrix} 5 & 3 & -2 \\ -2 & -6 & 4 \\ -1 & 1 & 2 \end{pmatrix}$$

old era

$\sigma = \frac{\text{mass}}{\text{Area}} = \frac{M}{8a^2}$ everything except x

#1 $I_{xx} = \sigma \int_{-2a}^{2a}\int_{-a}^{a} dx\,dy\,y^2 = \frac{M}{8a^2}\int_{-2a}^{2a} dx\,\frac{2a^3}{3} = \frac{M}{8a^2}\,4a\,\frac{2}{3}a^3 = \frac{1}{3}Ma^2$

$I_{yy} = \sigma \int_{-2a}^{2a}\int_{-a}^{a} dx\,dy\,x^2 = \frac{M}{8a^2}\,\frac{2}{3}(2a)^3\,2a = \frac{4}{3}Ma^2$

$I_{zz} = I_{xx} + I_{yy}$ ✓

$I_{xy} \sim \int dx\,dy\,xy$ ← each integral is zero

$\vec{L} = \frac{1}{3}Ma^2 \begin{pmatrix} 1 & & \\ & 4 & \\ & & 5 \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$ $\quad \omega_z = 0$, $\omega_x = 2\omega_y$

$= \frac{1}{3}Ma^2 \begin{pmatrix} \omega_x \\ 4\omega_y \\ 0 \end{pmatrix} = \frac{1}{3}Ma^2 \begin{pmatrix} 2\omega_y \\ 4\omega_y \\ 0 \end{pmatrix}$

b) $\vec{L}$ has an unchanging part along ω & a part that circles in a plane $\perp$ to $\vec{\omega}$ ($L_\perp$)
when plate rotates some small angle ϕ, $\Delta L_\perp = L_\perp \Delta\phi$

$\frac{\Delta L_\perp}{\Delta t} = L_T\,\dot{\phi} = L\sin\theta\,\omega = \overrightarrow{\omega \times L}$

$\Delta\vec{L}$ & Γ are in $\hat{k}$ direction $= \frac{1}{3}Ma^2 \begin{vmatrix} i & j & k \\ 2\omega_y & \omega_y & 0 \\ 2\omega_y & 4\omega_y & 0 \end{vmatrix}$

$= \frac{1}{3}Ma^2 \left| -2\omega_y^2 + 8\omega_y^2 \right| = \frac{6}{3}Ma^2\omega_y^2$
$= 2Ma^2\omega_y^2$

2 

putting things together: (inertial) $= M_\phi M_\theta M_\psi$ (body)

body axis is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow$ (inertial) $= M_\phi M_\theta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$= M_\phi \begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix}$

$= \begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}$

↑ usual $\hat{r}$ direction for ϕ, θ

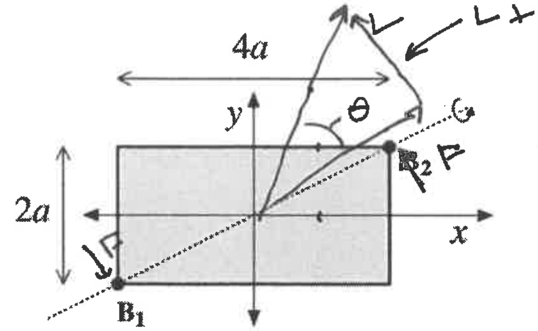
Complete 4 of these 5 problems

1. Consider a flat, uniform rectangular plate of mass: M, and with sides: $4a \times 2a$ that at this instant lies in the z plane.

   (a) The moment of inertia tensor of this plate (about its CM) is given by:

   $$I_P = \frac{1}{3}Ma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

   Calculate the integral that is I_{yy} to confirm the above entry. Explain why the integral that is I_{xy} is zero.



   (b) The plate is rotating at angular velocity ω about the (fixed) axis shown (a diagonal). Calculate the angular momentum (about its CM) at this instant. Draw the resulting vector directly on the diagram.

   (c) Report the direction and magnitude of the torque (about the CM) on the plate.

   (d) $\mathbf{B_1}$ and $\mathbf{B_2}$ are the bearings for the rotation axis. Directly on the diagram above, show the direction of any forces applied to the plate at those locations.

2. In the body-fixed frame the axes (123) are aligned with the principal axes and the body's symmetry axis is coincident with the 3-axis. We seek the direction of the body-fixed 3-axis in the inertial ('space') frame. At this instant, the orientation of the body-fixed frame is described by three Euler angles: ϕ, θ, ψ. The connection between the inertial frame and the body-fixed frame is made through three successive rotations: (1) a rotation about the z axis by the angle ϕ connecting the inertial frame to frame″ [(inertial) = $\mathcal{M}_\phi$·(frame″)]

$$\mathcal{M}_\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix} \; ?$$

(2) a rotation about the y'' axis by θ connecting frame″ to frame′ [(frame″) = $\mathcal{M}_\theta$·(frame′)]

$$\mathcal{M}_\theta = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

(3) a rotation about the z' axis by ψ connecting the frame′ to the body-fixed frame (123):

$$\mathcal{M}_\psi = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Find an expression for the direction of the body-fixed 3-axis in the inertial frame.