To show: $\mathcal{M}\,\vec{A}\times\vec{B} = (\mathcal{M}\vec{A})\times(\mathcal{M}\vec{B}) \quad \forall\, \vec{A},\vec{B}$

or $\quad \vec{A}\times\vec{B}_i = \mathcal{M}^{-1}\left[(\mathcal{M}\vec{A})\times(\mathcal{M}\vec{B})\right]_i$

True $\forall A B$ requires

$$\sum_{jk} \epsilon_{ijk} A_j B_k = \sum_{\substack{a,bc\\jk}} \mathcal{M}^T_{ia}\, \epsilon_{abc}\, \mathcal{M}_{bj} A_j\, \mathcal{M}_{ck} B_k$$

$$\epsilon_{ijk} = \sum_{abc} \epsilon_{abc}\, \underbrace{\mathcal{M}^T_{ia}\, \mathcal{M}^T_{jb}\, \mathcal{M}^T_{kc}}_{\text{if } ijk=123 \text{ this is exactly } \det(\mathcal{M}^T) = 1}$$

Levi-Civita symbol exactly same as $(-)^\sigma$
$= +1$ if ijk even # swaps of 123
$= -1$ if ijk odd # swaps of 123
$= 0$ if ijk not a permutation (eg has a repeat)

So: $\epsilon_{312} = +1$, $\epsilon_{132} = -1$, $\epsilon_{112} = 0$

if ijk has a repeated index is det of matrix with identical rows $\Rightarrow 0$

$$= \sum_{\sigma\in S_N} (-1)^\sigma\, \mathcal{M}^T_{i\sigma 1}\, \mathcal{M}^T_{j\sigma 2}\, \mathcal{M}^T_{k\sigma 3}$$

$\leftarrow$ what if ijk is a permutation of 123, say $i = P1$, $j = P2$, $k = P3$

$$= \sum_{\sigma\in S_N} (-1)^\sigma\, \mathcal{M}^T_{P1\,\sigma 1}\, \mathcal{M}^T_{P2\,\sigma 2}\, \mathcal{M}^T_{P3\,\sigma 3}$$

one of these row labels must be 1; $1 = Pj \Rightarrow j = P^{-1}1$

so $\mathcal{M}^T_{Pj\,\sigma j} = \mathcal{M}^T_{1\,\sigma P^{-1}1}$

$$= \sum_{\sigma\in S_N} (-)^\sigma\, \mathcal{M}^T_{1\,\sigma P^{-1}(1)}\, \mathcal{M}^T_{2\,\sigma P^{-1}2}\, \mathcal{M}^T_{3\,\sigma P^{-1}3}$$

summing over all σ results in σP^{-1} covering all S_N so the difference between this and a determinant is $(-1)^\sigma$ vs $(-1)^{\sigma P^{-1}}$ but $(-1)^\sigma = (-1)^{\sigma P^{-1}}(-)^{P^{-1}}$

$$= (-)^{P^{-1}} \det \mathcal{M}^T = (-)^{P^{-1}} \text{ as } \det \mathcal{M}^T = 1$$

To Show: dot products are scalars: $\vec{A}' \cdot \vec{B}' = \vec{A} \cdot \vec{B}$

$$\vec{A}' \cdot \vec{B}' = \vec{A}'^T \vec{B}' = (m\vec{A})^T (m\vec{B}) = \vec{A}^T m^T m \vec{B} = \vec{A}^T \vec{B}$$

Transform tensors eg $\vec{L} = \overleftrightarrow{I}\vec{\omega} \rightarrow \vec{L}' = \overleftrightarrow{I}'\vec{\omega}'$

$$m\vec{L} = m\overleftrightarrow{I}\underbrace{\vec{\omega}}_{m^{-1}\omega'} = m\overleftrightarrow{I}m^{-1}\omega' = (m\overleftrightarrow{I}m^T)\vec{\omega}'$$

How does this work in different dimensions?

$N=2$ ... we'll calculate using usual formula but in $N=2$ there is no z direction.

$$\overline{A \times B} = \begin{vmatrix} i & j & k \\ A_1 & A_2 & 0 \\ B_1 & B_2 & 0 \end{vmatrix} = (A_1B_2 - A_2B_1)\hat{k} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

with rotation $\rightarrow$ $m^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} m = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$

s: $\sin\theta$ $\quad c$: $\cos\theta$

$$= \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} -s & c \\ -c & -s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so in rotated frame we get $A_1' B_2' - A_2' B_1'$

ie "cross product" is invariant $\leftarrow$ a scalar.

if we have a mirror reflection transformation $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -x \\ y \end{pmatrix}$

this "scalar" changes sign unlike a normal scalar

"pseudo scalar"

Note: in 3d mirror reflection leaves $\overline{A \times B}$ unchanged whereas a normal vector would invert $\rightarrow$ "pseudovector" or "axial vector"

Back to $N=2$: $\overline{A \times B} = \sum_{i,j} \epsilon_{ij} A_i B_j$ [Note no "free" index so scalar]

$\epsilon_{12} = 1$

$\epsilon_{21} = -1$

$\epsilon_{11} = 0$

$\epsilon_{22} = 0$

$N=4$ (four dimensions) "$\vec{A}\times\vec{B}$" $= \sum_{k,l} \underbrace{\epsilon_{ijkl}}$ $A_k B_l$

2 "free" indices so is a antisymmetric tensor with 6 elements

"Relativity" says universe really has four dimensions with time as added dimension. In 3d we had rotations which mixed x, y, z; if time really is like space should be transformations that mix space & time ... "boosts" - galilean trans

$x' = x - vt$ → but Lorentz corrects to $x' = \gamma(x - \beta ct)$

$t' = t$ → but Lorentz corrects to $\frac{1}{\sqrt{1-\beta^2}}$ v/c

$t' = \gamma\left(t - \beta\frac{x}{c}\right)$ small & hard to measure but true

in 1903 easy now

OK - given that there are transformations ("boosts") that seem like rotations between space & time Does all of the "vector" mechanism follow in $N=4$?

The key eqn of "rotations" was $M^T = M^{-1}$ from which followed dot product invariance. Is there a "dot product" invariance in relativity?

Yes - from the constant speed of light in every frame. If a flash happens at $r=0, t=0$ & $r'=0, t'=0$ [the origins of the two frames coincide at the same moment the flash occurs] then $r^2 - c^2t^2 = 0$ & $r'^2 - c^2t'^2 = 0$

to get this minus into our dot product use complex #s

$\mathbb{X} = (\vec{r}, ict)$; $\mathbb{X}\cdot\mathbb{X} = \vec{r}^{\,2} + (ict)^2 = \vec{r}^{\,2} - c^2t^2$

Boost. $\mathbb{X}' = \begin{pmatrix} \gamma & & & +i\gamma\beta \\ & 1 & & \\ & & 1 & \\ -i\gamma\beta & & & \gamma \end{pmatrix} \cdot \mathbb{X}$

you can check that this is orthogonal matrix ie $M^T M = 1$

rotations $\left(\begin{array}{c:c} \text{normal } 3\times3 \text{ rotation matrix} & 0 \\ \hdashline 0 & 1 \end{array}\right)$ no mixing of space & time.

pure boost $\left(\begin{array}{c:c} \mathbb{R} & i \\ \hdashline i & \mathbb{R} \end{array}\right)$ Imaginary needed here so when mix time & space you get real or Imaginary as required

4 vectors denoted here with double bar

we can form new vectors by dividing $\mathbb{X}$ by an invariant (Δt Not invariant)

$$\Delta\tau^2 = \Delta t^2 - \frac{\Delta r^2}{c^2} = \Delta t^2\left(1 - \frac{v^2}{c^2}\right) \rightarrow d\tau = \frac{dt}{\gamma} \text{ is invariant}$$

$$\mathbb{U} = \frac{d\mathbb{X}}{d\tau} = \begin{pmatrix} \gamma \frac{d\vec{r}}{dt} \\ i\gamma c \end{pmatrix}$$

Note $\mathbb{U}\cdot\mathbb{U} = \gamma^2(v^2 - c^2)$

$$= -c^2\gamma^2\left(1 - \frac{v^2}{c^2}\right) = \tau^2$$

$$\mathbb{A} = \frac{d\mathbb{U}}{d\tau}$$

note since $\mathbb{U}\cdot\mathbb{U} = \text{const}$

$$\mathbb{U}\cdot\mathbb{A} = 0$$

momentum

invariant mass

relativistic momentum

$$\mathbb{P} = m\mathbb{U} = \begin{pmatrix} m\gamma\vec{v} \\ i\gamma mc \end{pmatrix} = \begin{pmatrix} m\gamma\vec{v} \\ i E/c \end{pmatrix}$$

cuz $E = \gamma mc^2$

Note: Law of conservation of momentum now automatically includes energy as the 4th component.

Note: you can make boost look a bit like rotation by using an "angle" called the rapidity ϕ

where $\left.\begin{aligned} \gamma &= \cosh\phi \\ \beta\gamma &= \sinh\phi \end{aligned}\right\}$ $\cosh^2\phi - \sinh^2\phi = 1$

$$\begin{pmatrix} \cosh\phi & +i\sinh\phi \\ -i\sinh\phi & \cosh\phi \end{pmatrix} \leftrightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\cos i\phi = \cosh\phi$$
$$\sin i\phi = i\sinh\phi$$