

To show: $M \bar{A} \times \bar{B} = (M \bar{A}) \wedge (M \bar{B}) \quad \forall \bar{A}, \bar{B}$

or $\bar{A} \times \bar{B}_i = M^T [(M \bar{A}) \wedge (M \bar{B})]_i$

True $\forall A, B$ requires $\sum_{j,k} \epsilon_{ijk} A_j B_k = \sum_{a,b,c} M^T_{ia} \epsilon_{abc} M_{bj} A_j M_{ck} B_k$

$\epsilon_{ijk} = \sum_{a,b,c} \epsilon_{abc} M^T_{ia} M^T_{jb} M^T_{kc}$

Levi-Civita symbol exactly same as $(-1)^{\sigma}$ if $ijk=123$ this is exactly $\det(M^T) = 1$

- = +1 if ijk even # swaps of 123
- = -1 if ijk odd # swaps of 123
- = 0 if ijk not a permutation (eg has a repeat)

So: $\epsilon_{312} = +1, \epsilon_{132} = -1, \epsilon_{112} = 0$

if ijk has a repeated index is det of matrix with identical rows $\Rightarrow 0$

= $\sum_{\sigma \in S_N} (-1)^{\sigma} M^T_{i\sigma_1} M^T_{j\sigma_2} M^T_{k\sigma_3}$ ← what if ijk is a permutation of 123

= $\sum_{\sigma \in S_N} (-1)^{\sigma} M^T_{p_1\sigma_1} M^T_{p_2\sigma_2} M^T_{p_3\sigma_3}$ say $i=p_1, j=p_2, k=p_3$

one of these row labels must be 1; $1 = p_j \Rightarrow j = p^{-1} 1$

so $M^T_{p_j \sigma_j} = M^T_{1 \sigma_{p^{-1} 1}}$

= $\sum_{\sigma \in S_N} (-1)^{\sigma} M^T_{1 \sigma_{p^{-1}(1)}} M^T_{2 \sigma_{p^{-1}(2)}} M^T_{3 \sigma_{p^{-1}(3)}}$

Summing over all σ results in $\sigma_{p^{-1}}$ covering all S_N so the difference between this and a determinant is $(-1)^{\sigma}$ vs $(-1)^{\sigma_{p^{-1}}}$ but $(-1)^{\sigma} = (-1)^{\sigma_{p^{-1}}} (-1)^{p^{-1}}$

= $(-1)^{p^{-1}} \det M^T = (-1)^{p^{-1}}$ as $\det M^T = 1$

To Show: dot products are scalars: $\vec{A}' \cdot \vec{B}' = \vec{A} \cdot \vec{B}$

$$\vec{A}' \cdot \vec{B}' = \vec{A}'^T \vec{B}' = (M\vec{A})^T (M\vec{B}) = \vec{A}^T M^T M \vec{B} = \vec{A}^T \vec{B}$$

Transform tensors e.g. $\vec{L} = \vec{I} \vec{w} \rightarrow \vec{L}' = \vec{I}' \vec{w}'$

$$M\vec{L} = M\vec{I}\vec{w} = M\vec{I}M^{-1}M\vec{w} = (M\vec{I}M^T)\vec{w}'$$

How does this work in different dimensions?

$N=2$... we'll calculate using usual formula but in $N=2$ there is no z direction.

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ A_1 & A_2 & 0 \\ B_1 & B_2 & 0 \end{vmatrix} = (A_1 B_2 - A_2 B_1) \hat{k} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

$$\text{with rotation} \rightarrow M^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

$\uparrow \sin\theta$ $\uparrow \cos\theta$

$$= \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} -s & c \\ -c & -s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so in rotated frame we get $A'_1 B'_2 - A'_2 B'_1$

ie "cross product" is invariant \leftarrow a scalar.

if we have a mirror reflection transformation $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -x \\ y \end{pmatrix}$

this "scalar" changes sign unlike a normal scalar
"pseudo scalar"

Note: in 3d mirror reflection leaves $\vec{A} \times \vec{B}$ unchanged

where as a normal vector would invert \rightarrow "pseudovector"
or "axial vector"

Back to $N=2$: $\vec{A} \times \vec{B} = \sum_{i,j} \epsilon_{ij} A_i B_j$ [Note no "free" index so scalar]

$$\begin{aligned} \epsilon_{12} &= 1 \\ \epsilon_{21} &= -1 \\ \epsilon_{11} &= 0 \\ \epsilon_{22} &= 0 \end{aligned}$$

$$N=4 \text{ (four dimensions)} \quad \overrightarrow{A \times B} = \sum_{k,l} \epsilon_{ijkl} A_k B_l$$

↳ 2 "free" indices
so is an antisymmetric
tensor with 6 elements

"Relativity" says universe really has
four dimensions with time as added dimension.

In 3d we had rotations which mixed x, y, z ; if
time really is like space should be transformations
that mix space & time ... "boosts" - galilean trans

$$x' = x - vt \rightarrow \text{but Lorentz corrects to } x' = \gamma(x - \beta ct)$$

$$t' = t \rightarrow \text{but Lorentz corrects to } \frac{1}{\sqrt{1-\beta^2}} \quad \uparrow \quad \uparrow \quad \uparrow$$

$$t' = \gamma(t - \beta \frac{x}{c}) \quad \text{small & hard to measure but true}$$

↳ in 1903 easy now

OK - given that there are transformations ("boosts")
that seem like rotations between space & time
Does all of the "vector" mechanics follow in $N=4$?

The key eqn of "rotations" was $M^T = M^{-1}$ from
which followed dot product invariance. Is there a
"dot product" invariance in relativity?

Yes - from the constant speed of light in every frame.

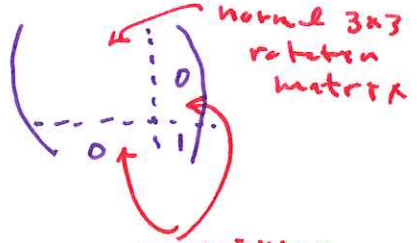
If a flash happens at $r=0, t=0$ & $r'=0, t'=0$ [the
origins of the two frames coincide at the same moment
the flash occurs] then $r^2 - c^2t^2 = 0$ & $r'^2 - c^2t'^2 = 0$

to get this minus into our dot product use complex #'s
 $X = (\vec{r}, ict)$; $X \cdot X = \vec{r}^2 + (ict)^2 = \vec{r}^2 - c^2t^2$

Boost.
$$X' = \begin{pmatrix} \gamma & & & +i\gamma\beta \\ & 1 & & \\ & & 1 & \\ -i\gamma\beta & & & \gamma \end{pmatrix} \cdot X$$

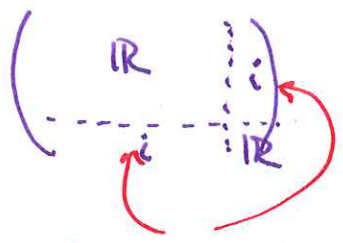
↳ you can check that this is orthogonal matrix,
ie $M^T M = I$

rotations



no mixing of space & time.

pure boost



imaginary needed here so when mix time & space you get real or imaginary as required

4 vectors denoted here with double bar

we can form new vectors by dividing X by an invariant (Δt not invariant)

$$\Delta \tau^2 = \Delta t^2 - \frac{\Delta r^2}{c^2} = \Delta t^2 \left(1 - \frac{v^2}{c^2}\right) \rightarrow d\tau = \frac{dt}{\gamma} \text{ is invariant}$$

$$U = \frac{dX}{d\tau} = \begin{pmatrix} \gamma \frac{d\vec{r}}{dt} \\ i\gamma c \end{pmatrix} \quad \text{Note } U \cdot U = \gamma^2 (v^2 - c^2) = -c^2 \gamma^2 \left(1 - \frac{v^2}{c^2}\right) = -c^2$$

$$A = \frac{dU}{d\tau} \quad \text{note since } U \cdot U = \text{const}$$

$$U \cdot A = 0$$

momentum \leftarrow invariant mass

$$P = mU = \begin{pmatrix} m\gamma \vec{v} \\ i\gamma mc \end{pmatrix} = \begin{pmatrix} m\gamma \vec{v} \\ iE/c \end{pmatrix} \quad \text{relativistic momentum} \quad \text{ cuz } E = \gamma mc^2$$

Note: Law of conservation of momentum now automatically includes energy as the 4th component.

Note: you can make boost look a bit like rotation by using an "angle" called the rapidity ϕ

$$\text{where } \left. \begin{aligned} \gamma &= \cosh \phi \\ \beta\gamma &= \sinh \phi \end{aligned} \right\} \cosh^2 \phi - \sinh^2 \phi = 1$$

$$\begin{pmatrix} \cosh \phi & +i \sinh \phi \\ -i \sinh \phi & \cosh \phi \end{pmatrix} \rightarrow \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

$$\begin{aligned} \cos i\phi &= \cosh \phi \\ \sin i\phi &= i \sinh \phi \end{aligned}$$