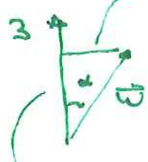
Discussion problem 10-43 — For a wobbling disk where in the body frame the angle between $\vec{\omega}$ & 3-axis is α, find the wobble frequency in the inertial frame. ← Eq 12 says is $\dfrac{I_3 \omega_3}{I_1 \cos\theta}$

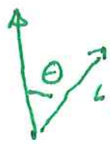
Given: Disk $\Rightarrow$ $\overleftrightarrow{I} = \dfrac{MR^2}{4}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}$

Eq 14:

$$\vec{\omega} = \left(\frac{p_\phi \sin\theta}{I_1} \sin\psi,\ \frac{p_\phi \sin\theta}{I_1} \cos\psi,\ \frac{p_\psi}{I_3} \right)$$



$p_\phi = \vec{L}$



$L_3 = p_\psi = L\cos\theta$

$$\tan\alpha = \frac{p_\phi \sin\theta / I_1}{p_\psi / I_3} = \overset{2}{\frac{I_3}{I_1}} \frac{p_\phi \sin\theta}{p_\phi \cos\theta} = 2\tan\theta$$

$$\frac{1}{\cos^2\theta} = 1 + \tan^2\theta = 1 + \left(\tfrac{1}{2}\tan\alpha\right)^2 \Rightarrow \frac{1}{\cos\theta} = \sqrt{1 + \tfrac{1}{4}\tan^2\alpha}$$

$$\omega_3 = \omega\cos\alpha$$

$$\dot{\phi} = \frac{I_3 \omega_3}{I_1 \cos\theta} = 2\,\omega\cos\alpha \sqrt{1 + \tfrac{1}{4}\tan^2\alpha} = \omega\sqrt{4\underbrace{\cos^2\alpha}_{1-\sin^2\alpha} + \sin^2\alpha}$$

$$= \omega\sqrt{4 - 3\sin^2\alpha}$$

[note $\dot{\phi} = 2\omega$ if α small] "sleeping top" – stable motion at $\theta = 0$ [ie gyro axis is vertical] suddenly leads to large nutation as top slows.

→ Cause: the effective potential $V = \dfrac{(b - a\cos\theta)^2}{2\sin^2\theta} + c^2\cos\theta$

goes from having a stable equilibrium at $\theta = 0$ to unstable. Note: $\theta = 0 \Rightarrow a = b$ as p_ψ is exactly the same thing as p_ϕ if gyro is vertical

$$\Rightarrow V = \frac{(1-\cos\theta)^2 a^2}{2\sin^2\theta} + c^2\cos\theta$$

Consider $\dfrac{V}{a^2} = \dfrac{(1-\cos\theta)^2}{2\sin^2\theta} + \dfrac{c^2}{a^2}\cos\theta$; Taylor expand

$\cos\theta \simeq 1 - \dfrac{\theta^2}{2} + \cdots$

$\sin\theta \simeq \theta - \underbrace{\dfrac{\theta^3}{3!}}_{\text{ignore}} + \cdots$

$$\frac{\overbrace{\left(1 - \left(1 - \frac{\theta^2}{2}\right)\right)^2}^{\frac{1}{4}\theta^4}}{2\theta^2} + \frac{c^2}{a^2}\left(1 - \frac{1}{2}\theta^2\right)$$

$$= \frac{1}{8}\theta^2 + \frac{c^2}{a^2} - \frac{1}{2}\frac{c^2}{a^2}\theta^2$$

$$= \frac{c^2}{a^2} + \frac{1}{2}\underbrace{\left(\frac{1}{4} - \frac{c^2}{a^2}\right)}\theta^2$$

with fast spin (large a) this is $(+)$ $\Rightarrow$

with slow spin this quantity is $(-)$ $\Rightarrow$

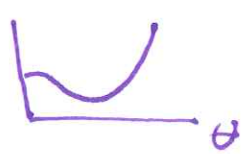
unstable

$$\frac{c^2}{a^2} = \frac{mgR/I_1}{(I_3\omega_3/I_1)^2}$$

$$= \frac{I_1 MgR}{I_3^2\omega_3^2} < \frac{1}{4}$$

↑ for stable motion.

FYI: Mathematica happy to return more Taylor terms shows next term is $+\theta^4$ so unstable case looks like:



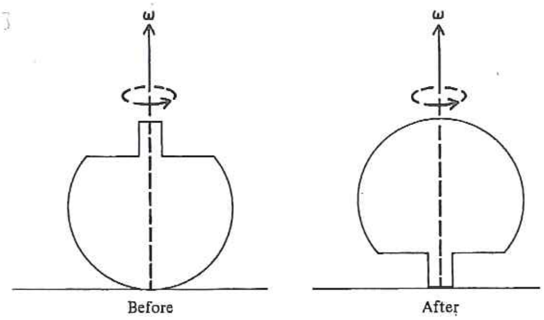
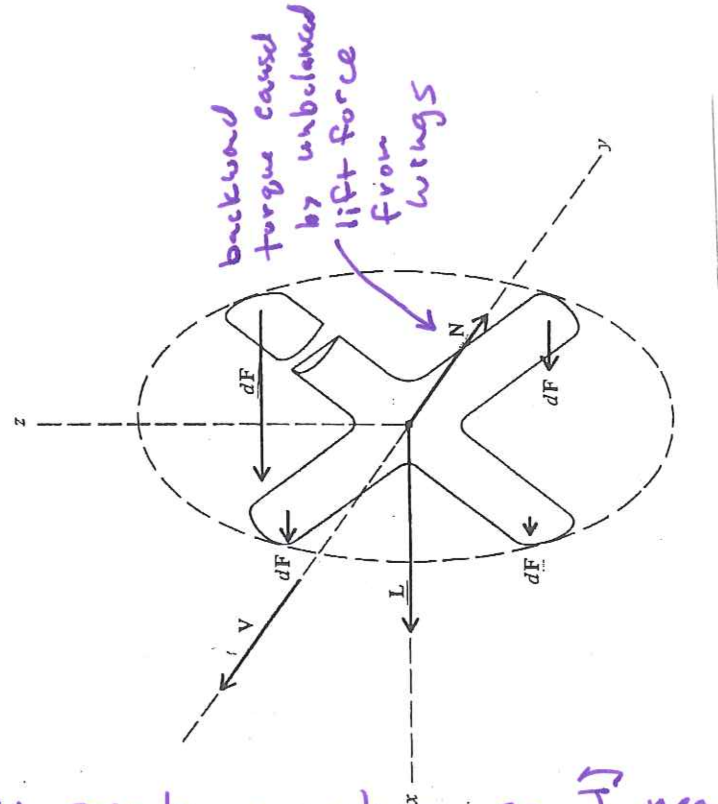


FIGURE 7-24. Flipping of a tippie-top.



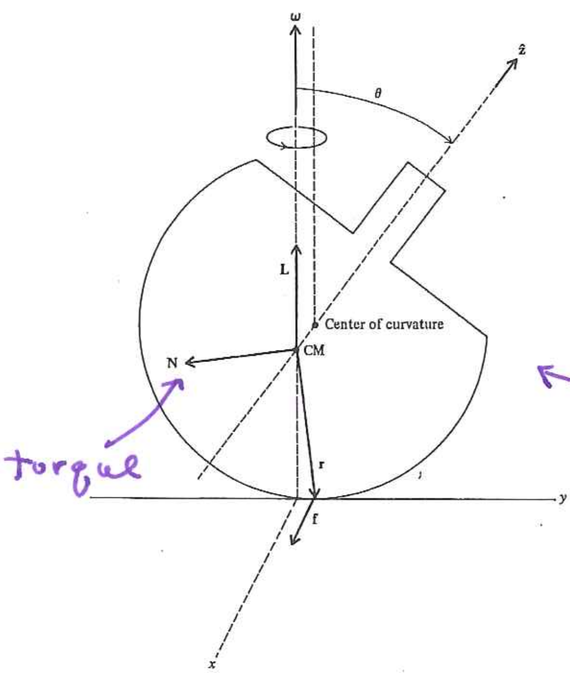
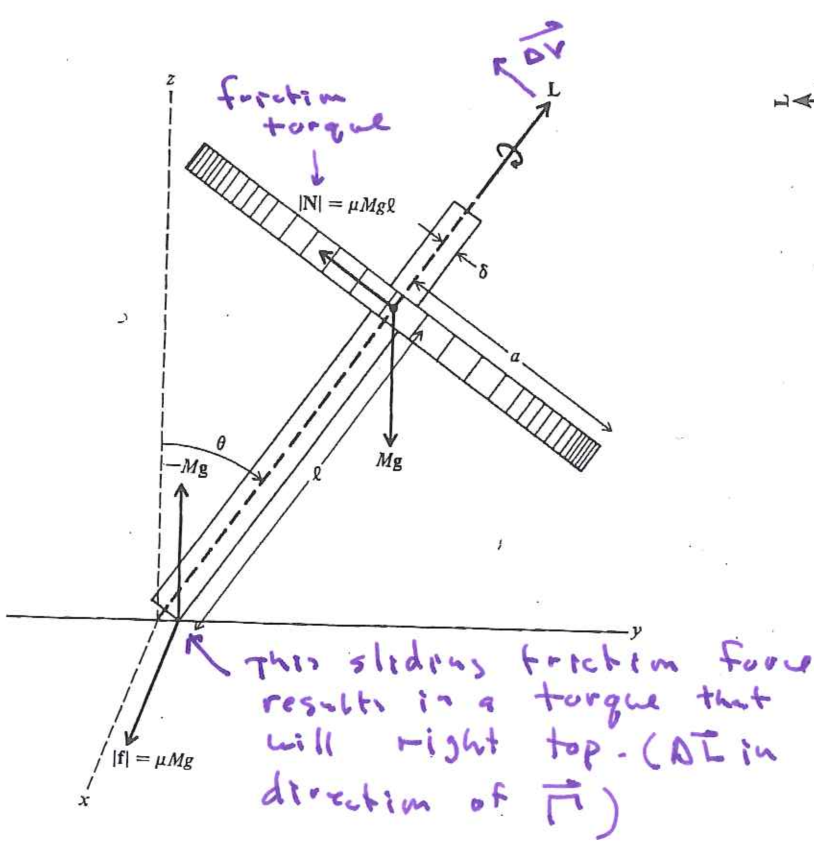


FIGURE 7-25. Frictional force and torque on a tippie-top.

top is nearly a sphere so $\overleftrightarrow{I}$ nearly diagonal & $\vec{L} \parallel \vec{\omega} \Rightarrow \overrightarrow{\omega \times L} = 0$
CM close to center so torque of gravity small compared to friction.
work in body frame; note $\vec{L} \perp$ torque



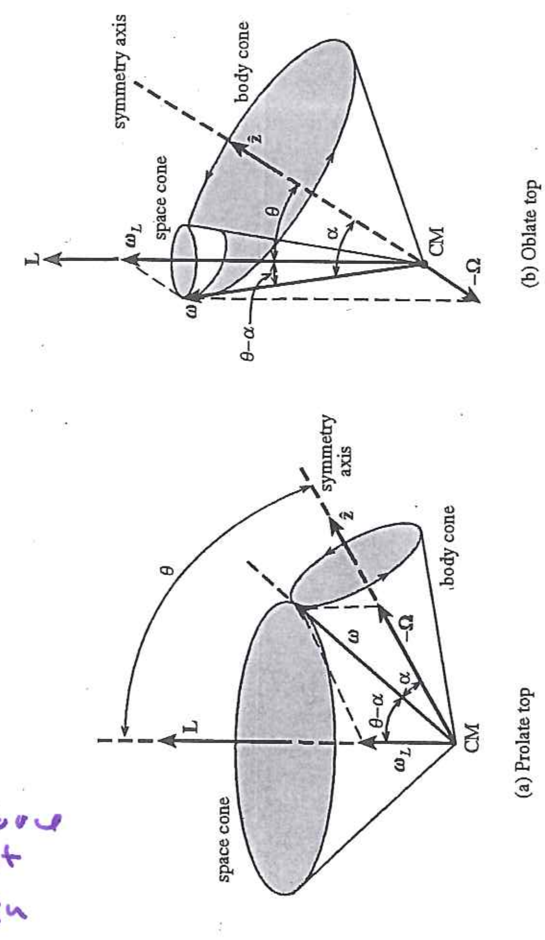


FIGURE 7-19. Space and body cones for (a) prolate top, (b) oblate top.

Earth as top: Equatorial bulge ($I_3 > I_1$)

gravitational torque tries to align equator with plane of Moons orbit

Result: Spin axis of Earth is not fixed — rather slow (26,000 year) precession.
Chandler wobble is now seen as nutation with backward loops ... in the case of weak external torques nutation of gyro = free precession of zero torque object.

In the past (eg Aristotle) there was no "Polaris" [north pole] star.

Note: "gyro" is perhaps the wrong word for what we've been talking about — "top" would be better.
Real gyros are designed to have zero torque — so conservation of $\vec{L}$ means they point in a fixed direction no matter how the case is moved
"gyro compass" are gyros that are constrained to rotated about a vertical axis — As per previous HW problems — Coriolis results in a torque that makes them point true north. (in contrast to magnetic north).