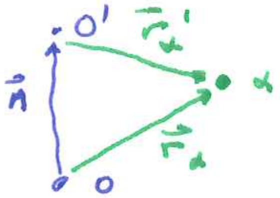
Seeking meaning of parallel axis Thm: $I = I_{cm} + Mh^2$
as if CM is not on axis $\vec{L}$ (in general) is not aligned with $\vec{\omega}$

Thm: The component of $\vec{L}$ in the direction of an axis is not changed if the origin is shifted along that axis
Ie the origin O' is shifted from O by vector $\vec{A}$
then $\vec{L}' \cdot \vec{A} = \vec{L} \cdot \vec{A}$

Main point: if $\vec{L}$ depends on origin (as it will if CM not on axis) a relationship like $\vec{L} = I\vec{\omega}$ is impossible.
However if we restrict to component of $\vec{L}$ along axis [take this to be L_z] & restrict to origins on that axis then $L_z = I\omega$ is a possible relationship.

Pf:



$$\vec{L}' = \sum m_\alpha \vec{r}_\alpha' \times \vec{v}_\alpha$$

$$\vec{L} = \sum m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha = \sum m_\alpha (\vec{A} + \vec{r}_\alpha') \times \vec{v}_\alpha$$

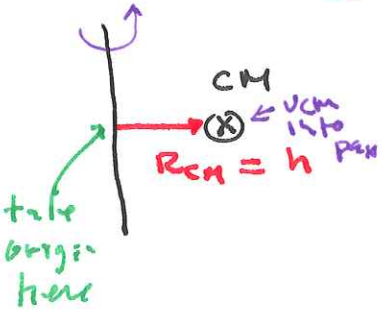
$$= \underbrace{\vec{A} \times \sum m_\alpha \vec{v}_\alpha} + \vec{L}'$$

but this term is ⊥ to $\vec{A}$ so when dotted with $\vec{A} \Rightarrow 0$

Now in the case of CM a distance h from axis

$$\vec{L} = \underbrace{M\vec{R}_{cm} \times \vec{V}_{cm}} + \vec{L}_{CM}$$

this is $I_{cm}\omega$



This piece depends on origin, but every origin on axis gives an $\vec{L}$ with the same component of $\vec{L}$ along that axis. For convenience we take origin exactly along side of $\vec{R}_{cm}$

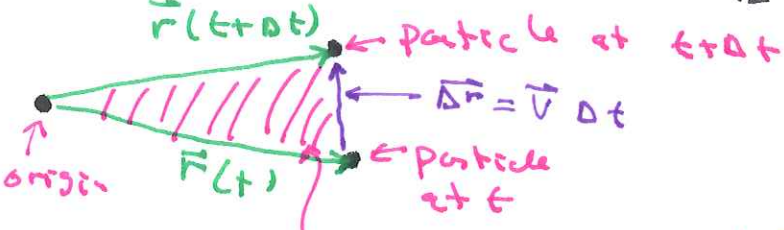
$\vec{V}_{cm}$ into page with magnitude $\omega R_{cm} = \omega h$
$\vec{R}_{cm}$ & $\vec{V}_{cm}$ are ⊥ so $\vec{R}_{cm} \times \vec{V}_{cm}$ has magnitude $|R_{cm}||V_{cm}| = h^2\omega$ and direction right up axis

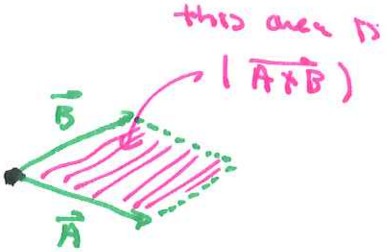
So $L_z = Mh^2\omega + I_{cm}\omega = (\underbrace{I_{cm} + Mh^2}_{\text{value of } I \text{ for this axis}})\omega$

Other misc issues in Chapter 3

→ Areal velocity = Consider a line connecting origin to particle. As particle moves that line will "sweep" an area.

$$\text{Areal velocity} = \frac{\underline{\text{area swept during } \Delta t}}{\Delta t}$$





this area is $\frac{1}{2}|\vec{r}(t) \times \underbrace{\vec{r}(t+\Delta t)}_{=\vec{r}(t)+\vec{v}\,\Delta t}|$

$$= \frac{1}{2}|\vec{r} \times \vec{v}\,\Delta t| \quad (\text{since } \vec{r} \times \vec{r} = 0)$$

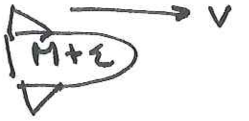
$$= \frac{1}{2m}|\vec{\ell}|\,\Delta t$$

So areal velocity $= \frac{1}{2m}|\vec{\ell}|$

→ Rocket Equation via Conservation of Momentum

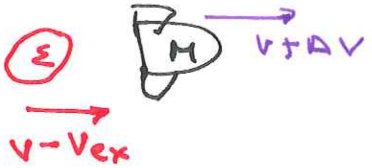
Consider rocket with No <u>external</u> force (eg in deep space) but using its engine. During some time Δt it exhausts a mass ε of burned fuel at some relative speed V_{ex}. Big <u>internal</u> forces between burned fuel & rocket but No external force — total momentum conserved

initial:



initially moving at some speed V

final:



Note: mass of rocket has been reduced $dM = -\varepsilon$

$$(M+\varepsilon)\,V = M(V+\Delta V) + \varepsilon(V-V_{ex})$$

$$0 = M\,\Delta V - \varepsilon V_{ex} = M\,dv + dM\,v_{ex} +$$

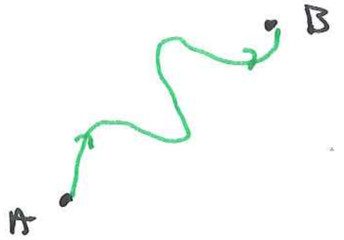
diff eq: $-v_{ex} = M\frac{dv}{dM} \longrightarrow$ solution $V_{ex}\ln\left(\frac{M_i}{M_f}\right) = V_f - V_i$

"thrust" $= -V_{ex}\frac{dM}{dt} = M\frac{dv}{dt}$

Kinetic Energy for one Particle $= \frac{1}{2}mv^2 = \frac{1}{2}m\vec{v}\cdot\vec{v} = T$

$$\frac{dT}{dt} = \underbrace{m\vec{a}}_{\text{total force } F \text{ on particle}}\cdot\vec{v} = \vec{F}\cdot\vec{v}$$

← Called power in 191; unit watts



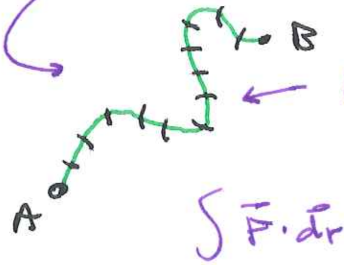
Consider change in KE as particle moves from A → B:

$$T_B - T_A = \int_{t_A}^{t_B} \frac{dT}{dt}\,dt = \int_{t_A}^{t_B} \underbrace{\vec{F}\cdot\vec{v}}_{\text{note: the force \& velocity will depend on time}}\,dt$$

$$= \underbrace{\int \overbrace{\vec{F}\cdot d\vec{r}}^{\text{Work unit Joules}}}_{\text{line integral}}$$

← Since $\vec{v} = \frac{d\vec{r}}{dt}$

How to calculate a line integral



← break up path into little steps $\vec{\Delta r}$

$$\int \vec{F}\cdot d\vec{r} = \sum \vec{F}\cdot\vec{\Delta r}$$

add up those steps!
(and take limit as stepsize → 0)

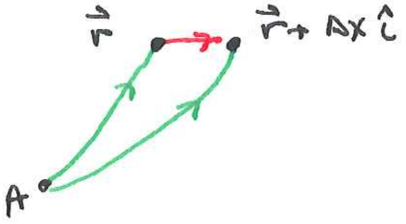
If the force just depends on position it should now be clear that the line integral does not depend on the speed used on the path. Does the integral depend on the exact path or just the begin/end points? Surprisingly for many forces the path integral is independent of path – Forces for which that is the case are called Conservative Forces. Note: friction is not a conservative force.

Important: The sum of the work of all applied Forces = change in KE

If the force is Conservative [so line integral only depends on end points] we can define a function which gives the integral for any end point

$$\phi(\vec r) = -\int_A^{\vec r} \vec F\cdot dr$$

[This turns out to be PE]



$$\phi(\vec r+\Delta x\hat\imath) - \phi(\vec r) = \text{red line integral}$$

$$= \int_{\vec r}^{\vec r+\Delta x\hat\imath} -\vec F\cdot d\vec r$$

$$\approx -\vec F\cdot \overset{\Delta x\hat\imath}{\Delta\vec r} = -F_x\,\Delta x$$

So: $$\frac{\partial\phi}{\partial x} = \lim_{\Delta x\to 0}\frac{\phi(\vec r+\Delta x\hat\imath)-\phi(\vec r)}{\Delta x} = -F_x$$

so $$\vec F = \left(-\frac{\partial\phi}{\partial x},\ -\frac{\partial\phi}{\partial y},\ -\frac{\partial\phi}{\partial z}\right) \equiv -\underset{\text{grad}}{\vec\nabla}\,\phi$$

For future use we define:

$$\underset{\text{div}}{\vec\nabla}\cdot\vec F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\underset{\text{curl}}{\vec\nabla}\times\vec F = \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \hat\imath\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) \text{ etc}$$

Note: $\vec\nabla\times\vec\nabla\phi = 0$ as involves: $\hat\imath\left(\frac{\partial}{\partial y}\frac{\partial}{\partial z}\phi - \frac{\partial}{\partial z}\frac{\partial}{\partial y}\phi\right)$ etc

Important. $\vec\nabla\times\vec F = 0 \Rightarrow \int\vec F\cdot dr$ is independent of Path and $\phi(\vec r) = -\int_A^{\vec r}\vec F\,dr$ is well defined

Pf involves Stokes Thm which you should see in your Multi Calc class