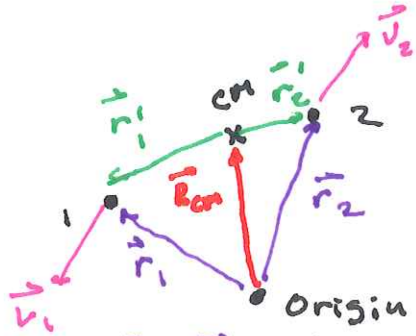
A composite system with mass M_A & CM $\vec{R}_A$; same for B

Combined CM : $\vec{R}_{cm} = \dfrac{M_A \vec{R}_A + M_B \vec{R}_B}{M_A + M_B}$ ie treat composites as if point masses



$\vec{r}_\alpha = \vec{R}_{cm} + \vec{r}_\alpha'$

$\vec{r}_\alpha'$: Coordinates relative to CM

$\vec{v}_\alpha'$: velocity relative to CM

Claim: $\sum m_\alpha \vec{r}_\alpha' = 0$ [therefore $\sum m_\alpha \vec{v}_\alpha' = 0$]

Pf. $\cancel{\vec{R}_{cm}} = \dfrac{\sum m_\alpha \vec{r}_\alpha}{M} = \dfrac{\sum m_\alpha (\vec{R}_{cm} + \vec{r}_\alpha')}{M}$

$= \dfrac{\sum m_\alpha \vec{R}_{cm}}{M} + \dfrac{1}{M}\sum m_\alpha \vec{r}_\alpha'$

$= \cancel{\vec{R}_{cm}} + \dfrac{1}{M}\sum m_\alpha \vec{r}_\alpha'$

$0 = \sum m_\alpha \vec{r}_\alpha' \xleftarrow{\text{so}}$

Total Angular Momentum $= \sum \vec{r}_\alpha \times m_\alpha \vec{v}_\alpha = \sum m_\alpha (\vec{r}_\alpha' + \vec{R}_{cm}) \times (\vec{v}_\alpha' + \vec{V}_{cm})$

$= \underset{\substack{\text{"about CM"} \\ \text{spin}}}{\sum m_\alpha \vec{r}_\alpha' \times \vec{v}_\alpha'} + \underset{\substack{\text{"of CM"} \\ \text{orbital}}}{M \vec{R}_{cm} \times \vec{V}_{cm}} + \underbrace{\sum m_\alpha \vec{r}_\alpha' \times \vec{V}_{cm}}_{\text{zero}} + \underbrace{\vec{R}_{cm} \times \sum m_\alpha \vec{v}_\alpha'}_{\text{zero}}$

Note: spin angular momentum does not depend on origin choice

Seeking the 1d version of 'moment of inertia' I : $L = I\omega$

(Remark: In chapter 10 I becomes a 3x3 matrix)

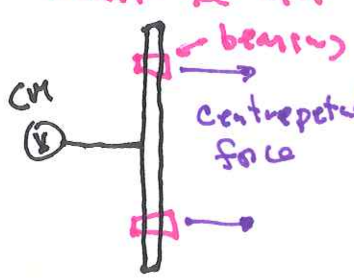
Required for a simple relationship: $\vec{L} = I\vec{\omega}$:

(A) $\vec{L}$ must be independent of origin [ie pure spin]

(B) $\vec{L}$ must be in same direction as $\vec{\omega}$

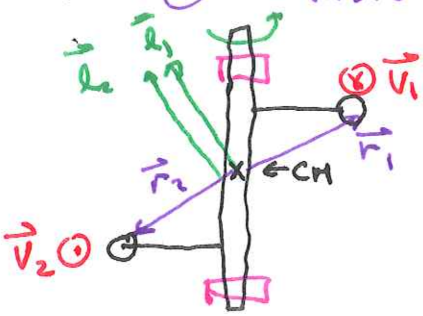
Achieve (A) by having the CM on the axis of rotation

Note if CM is off axis it is moving in a circle and hence accelerating. If the CM is accelerating there must be an external force (from axel bearings)

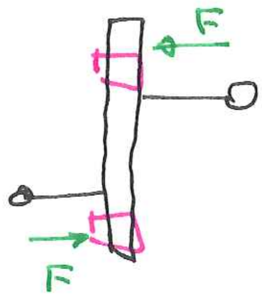


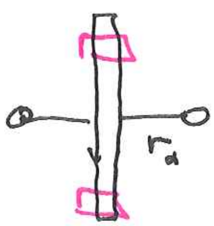
"static balance" of tire: make sure CM on spin axis

RE: (15) Consider system as shown — Note CM is on axis



Follow thru below diagram and discover $\vec{L} = \vec{\ell}_1 + \vec{\ell}_2$ is NOT in direction of axel. As object rotates $\vec{L}$ changes direction with object — $\Delta \vec{L}$ is out of page. Since $\frac{d\vec{L}}{dt} \neq 0$ there must be a torque supplied by bearings. Design system so bearings do Not have to provide these forces "dynamic balance". Avoid this problem by having symmetric object





$$v_\alpha = r_\alpha \omega$$

$$L = \underbrace{\sum m_\alpha r_\alpha^2}_{I}\, \omega$$

Note: this r_α is distance to axel NOT distance to origin

Continuum Approximation: Stuff really is made out of point particles (electrons & quarks) but approx as if mass evenly distributed: $dM = \rho\, dV$

mass of piece — density — volume of piece

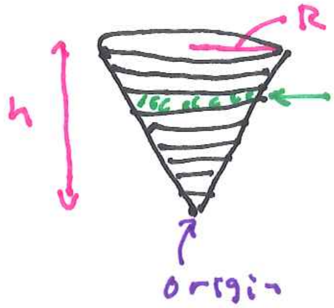
or in the case of sheets: $dM = \sigma\, dA$

$\frac{\text{mass}}{\text{Area}}$ — area of piece

Result: $\sum_\alpha \rightarrow \int dV$

Basic Idea — break object into small pieces whose properties (CM, I) you know. Riemann Sum of pieces becomes integral

CM of Cone:



CM of this disk is at center - altitude z

$\text{Mass} = \underbrace{\pi r^2 \Delta z}_{\text{volume}}\, \rho$

$$z_{CM} = \frac{1}{M} \sum \underbrace{\pi r^2 \Delta z \rho}_{\text{mass}} \underbrace{z}_{\text{Location}}$$

Note: $r = \frac{R}{h} z$

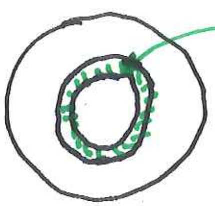
$$M = \rho V = \rho \frac{1}{3} \pi R^2 h$$

$$= \frac{1}{M} \int_0^h \pi \left(\frac{R}{h} z\right)^2 z \rho \, dz$$

$$= \frac{\cancel{\pi} \frac{\cancel{R^2}}{h^2} \cancel{\rho} \int_0^h z^3 dz}{\cancel{\rho} \frac{1}{3} \cancel{\pi} \cancel{R^2} h} = \frac{\frac{1}{4} h^4}{\frac{1}{3} h^3}$$

$= \frac{3}{4} h$ ← check that units make sense!

to find I of cone need I of disk



mass of annulus $2\pi r \Delta r \, \sigma$

distance from axis $= r$

$$\sum 2\pi r \Delta r \, \sigma \, r^2 = 2\pi\sigma \int_0^R r^3 dr = 2\pi\sigma \frac{R^4}{4} = \frac{1}{2} (\pi R^2 \sigma) R^2$$

$$= \frac{1}{2} M R^2$$

Now find I of cone by summing I's of thin disks

$$I = \sum \frac{1}{2} (\rho \pi r^2 \Delta z) r^2 = \frac{1}{2} \rho \pi \int_0^h \left(\frac{R}{h} z\right)^4 dz$$

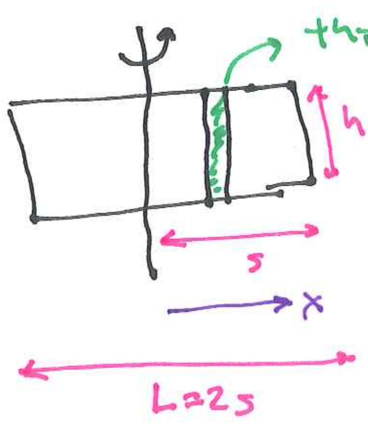
$r = \frac{R}{h} z$

$$= \frac{1}{2} \rho \pi \frac{R^4}{h^4} \int_0^h z^4 dz = \frac{1}{2} \rho \pi \frac{R^4}{h^4} \frac{h^5}{5} = \frac{3}{2 \cdot 5} \left(\rho \frac{1}{3} \pi R^2 h\right) R^2$$

$$= \frac{3}{10} M R^2$$

I of sheet (rectangle) rotated thru CM

thin piece: mass $= \sigma h dx$; distance from axis $= x$



$$I = 2 \int_0^s (\sigma h dx) x^2 = 2\sigma h \int_0^s x^2 dx$$

each half is same

$$= \frac{2}{3} \sigma h s^3 = \frac{1}{3} (\sigma h 2s) s^2$$

$$= \frac{1}{3} M s^2 = \frac{1}{12} M L^2$$

$L = 2s$

I of sphere - slice into disk - each one $I = \frac{1}{2} M r^2$

← This disk: Mass $= \rho \pi r^2 dz$

$$I = \frac{1}{2} \rho \pi r^2 dz\, r^2$$

Note: $z^2 + r^2 = R^2 \rightarrow r^2 = R^2 - z^2$

$$I = 2 \int_0^R \frac{1}{2} \rho \pi (R^2 - z^2)^2 \, dz = \rho \pi \int_0^R (R^4 - 2R^2 z^2 + z^4) dz$$

each hemisphere same

$$= \rho \pi \left[\frac{R^5}{1} - 2R^2 \frac{R^3}{3} + \frac{R^5}{5} \right]$$

$$= \rho \pi R^5 \left[1 - \frac{2}{3} + \frac{1}{5} \right] = \frac{8}{15}$$

$$= \left(\rho \frac{4}{3} \pi R^3 \right) R^2 \frac{2}{5} = \frac{2}{5} M R^2$$

Parallel Axis Thm: If you do not have the CM on the axis then previous conditions are NOT satisfied & bearing must provide force. [$\vec{L}$ will depend on origin and in general will not be aligned with axel] But for origins on the axel (call this the z axis) L_z will be independent of which point on z axis you select as origin. L_z will be given by

$$L_z = (I_{cm} + M h^2) \omega$$

I thru a parallel axis but one that goes thru CM

total mass

distance CM is from axel