Problem: charged particle (q) in a uniform magnetic field $\vec{B}$

$$m\dot{\vec{v}} = q\,\vec{v}\times\vec{B}$$

Note: speed $v = \sqrt{\vec{v}\cdot\vec{v}}$ is constant — Proof

$$\frac{d}{dt}(\vec{v}\cdot\vec{v}) = 2\vec{v}\cdot\vec{a} = 2\vec{v}\cdot\left(\frac{q}{m}\vec{v}\times\vec{B}\right) = 0 \leftarrow \text{as } \vec{v}\times\vec{B}\perp\vec{v}$$

Note: velocity in direction of $\vec{B}$ is constant — Proof

$$\frac{d}{dt}(\vec{v}\cdot\vec{B}) = \vec{a}\cdot\vec{B} = \frac{q}{m}(v\times B)\cdot B = 0 \leftarrow \text{as } \vec{v}\times\vec{B}\perp\vec{B}$$

To be specific lets take $\vec{B}$ in $\hat{z}$ direction so

$v_z = $ constant. Let $\vec{v}_\perp = (v_x, v_y, 0)$ — so $|\vec{v}_\perp| = $ constant

$$\dot{\vec{v}}_\perp = \frac{qB}{m}\vec{v}_\perp\times\hat{k} = \frac{qB}{m}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ v_x & v_y & 0\\ 0 & 0 & 1\end{vmatrix} = \underbrace{\frac{qB}{m}}_{\text{call this } \omega}\begin{pmatrix}v_y\\ -v_x\\ 0\end{pmatrix}$$

$$\dot{\vec{v}}_\perp = \begin{pmatrix}\dot{v}_x\\ \dot{v}_y\\ 0\end{pmatrix}$$

So... coupled 1st order diff eq

$$\left.\begin{aligned}\dot{v}_x &= \omega v_y\\ \dot{v}_y &= -\omega v_x\end{aligned}\right\}$$

text decides to solve this via complex #s as a way to start using Eulers Equation: $e^{i\theta} = \cos\theta + i\sin\theta$

I take a different approach.

write as matrix eq:

$$\begin{pmatrix}\dot{v}_x\\ \dot{v}_y\end{pmatrix} = \begin{pmatrix}0 & \omega\\ -\omega & 0\end{pmatrix}\begin{pmatrix}v_x\\ v_y\end{pmatrix}$$

$$\dot{v}_\perp = M v_\perp$$

If these were not vectors you'd immediately write solution: $v_\perp = e^{Mt}A$

THIS ALSO WORKS FOR MATRIX!

$$\begin{pmatrix}v_x\\ v_y\end{pmatrix} = e^{\begin{pmatrix}0 & \omega\\ -\omega & 0\end{pmatrix}t}\begin{pmatrix}v_{x0}\\ v_{y0}\end{pmatrix}$$

e^{Matrix} may sound strange but it in fact works out usually

Dull alternative: convert 2 first order diff eq to 1 second order

$$\ddot{v}_x = \omega\dot{v}_y = -\omega^2 v_x \longrightarrow v_x = A\sin(\omega t+\phi)$$

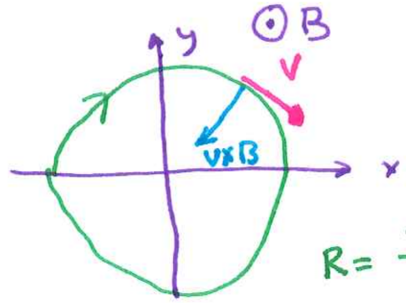
$$v_y = \frac{1}{\omega}\dot{v}_x = A\cos(\omega t+\phi)$$

Note: $v_x^2 + v_y^2 = $ constant as required

$$\frac{dx}{dt} = A\sin(\omega t+\phi) \longrightarrow x-x_0 = -\frac{A}{\omega}\cos(\omega t+\phi)$$

$$\frac{dy}{dt} = A\cos(\omega t+\phi) \longrightarrow y-y_0 = \frac{A}{\omega}\sin(\omega t+\phi)$$

Note $\Delta x^2+\Delta y^2 = \frac{A^2}{\omega^2}$ So motion is on a circle centered on x_0, y_0

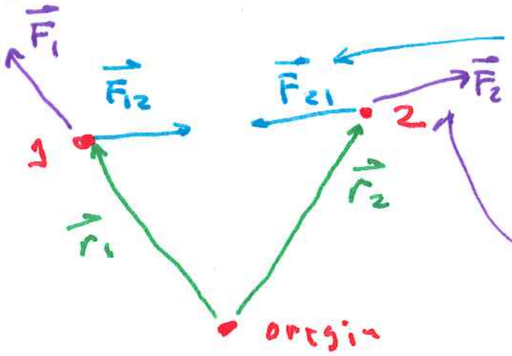


$R = \frac{v_0}{\omega}$

Recall: uniform z motion so full path is helix

Systems of Particles: α labels each of N particles

Mostly we'll write about just 2 particles but see that proofs apply to N particles.



equal opposite forces between particles in our "system"
[Note: Newton's 3RD Law]

external force (ie not caused by some particle in "system")

net force on (1): $\vec F_1 + \vec F_{12} = m_1\dot{\vec v}_1$ — add to zero!

\+ net force on (2): $\vec F_2 + \vec F_{21} = m_2\dot{\vec v}_2$

total external force $\vec F_1 + \vec F_2 = \frac{d}{dt}\underbrace{(m_1\vec v_1 + m_2\vec v_2)}_{\text{total momentum } \vec P}$

If we try to write total momentum $\vec P = M\vec V$ ← total mass

$\vec V = \frac{1}{M}(m_1\vec v_1 + m_2\vec v_2)$ ← velocity of center of mass

$\vec R = \frac{1}{M}(m_1\vec r_1 + m_2\vec r_2)$ ← location of center of mass

Note: if (total external force) = 0 then $\vec V_{cm}$ = Constant

if, in addition, initial $\vec V_{cm} = 0$ then $\vec V_{cm} = 0$ always

and $\vec R_{cm}$ = Constant → "canoe problems"

Angular Momentum of a particle $\vec{l} = \vec{r} \times m\vec{v} = \vec{r} \times \vec{p}$

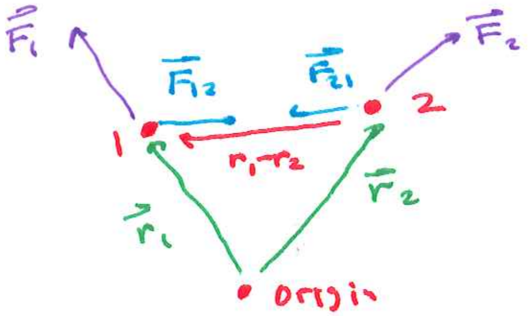
Note: in polar coordinates $\vec{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$ so $\vec{l} = rm r\dot{\phi}\, \underbrace{\hat{r} \times \hat{\phi}}_{\hat{z}}$

$= mr^2\dot{\phi}\,\hat{z}$

$mr^2 \leftarrow$ see "moment of Inertia"

$$\frac{d\vec{l}}{dt} = \frac{d}{dt}(\vec{r} \times m\vec{v}) = \overbrace{\vec{v} \times m\vec{v}}^{\text{zero}} + \vec{r} \times m\vec{a}$$

$$= \vec{r} \times \vec{F} = \text{torque } \Gamma$$



total angular momentum: add up angular momentum of every particle in system $\vec{L} = \sum_\alpha \vec{l}_\alpha$

Time derivative of total angular momentum

$$\frac{d\vec{L}}{dt} = \sum \vec{r}_\alpha \times \vec{F}_\alpha$$

sum of external & internal

$$= \vec{r}_1 \times (\vec{F}_1 + \vec{F}_{12}) + \vec{r}_2 \times (\vec{F}_2 + \vec{F}_{21})$$

$$= \underbrace{\vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2}_{\text{total external torque}} + \underbrace{(\vec{r}_1 - \vec{r}_2) \times \vec{F}_{12}}_{\text{zero}}$$

Note: net external Force $= 0 \Rightarrow$ total momentum conserved

net external Torque $= 0 \Rightarrow$ total angular momentum conserved