Foucault Pendulum –

– need only consider vertical component of $\Omega \Rightarrow \Omega \sin\lambda$
as the horizontal component of $\Omega \Rightarrow$ zero force
(if motion NS or $\underbrace{\text{vertical force}}$ if motion EW)

changes in vertical force result
only in changes in Tension.

– we consider pendulum in small angle approx –
thus the motion is effectively in a plane
The "restoring force" $mg\sin\theta \approx mg\,\theta \approx mg\frac{s}{R}$

so $\vec{F} = -\frac{mg}{R}\vec{r}$; $\frac{g}{R} = \omega_0^2$ in following

– Diff Eq:

$$\ddot{x} = -\omega_0^2 x + 2\Omega\dot{y}$$
$$\ddot{y} = -\omega_0^2 y - 2\Omega\dot{x}$$

From here on
Ω = vertical component of $\vec{\Omega}$
$= \Omega \sin\lambda$

$\Omega = \frac{2\pi}{24\cdot 3600}$; λ = latitude

→ This Diff Eq is linear
so superposition applies

→ It helps to put (x,y) vector
into $z = x+iy$ complex number.

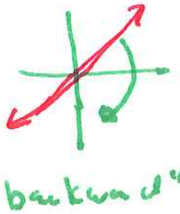
Oscillation in a fixed direction is then

$z = z_0 \cos(\ t)$ ← the vector z_0 stretched & shrunk by $\cos(\ t)$

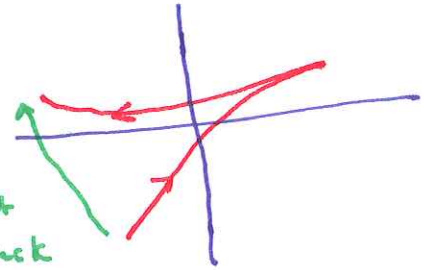
→ If plane of oscillation gradually changes:

$$z = e^{-i\Omega t}\cos(\omega_0 t)$$

Note: Coriolis Force "to right"



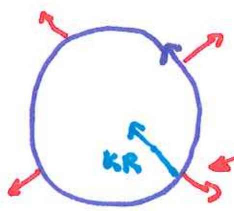
"backward" motion of back/forth plane



slow drift of back forth plane

Foucault Pendulum I – Consider conical motion
(For notational ease spring force $-k\vec{r}$ replaces gravity $mg\sin\theta$)

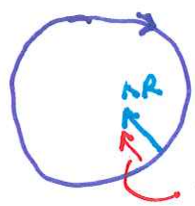
Ω ⊙



$2m\Omega v$ outward
ωr where ω = angular velocity of conic pendulum

$$KR - 2m\Omega\omega R = m\omega^2 R$$

$$\frac{K}{m} = \omega_0^2 = \omega^2 + 2\omega\Omega \rightarrow \omega_1 = \omega_0 - \Omega \text{ approx}$$



$2m\Omega\omega R$

$$KR + 2m\omega\Omega R = m\omega^2 R$$

$$\frac{K}{m} = \omega_0^2 = \omega^2 - 2\omega\Omega \rightarrow \omega_2 = \omega_0 + \Omega \text{ approx}$$

Note: $\left.\begin{array}{l}\omega_0 \rightarrow \text{period of a few seconds} \\ \Omega \rightarrow \text{period of a day}\end{array}\right\} \Omega \ll \omega_0$

Given the very small difference in periods – how detect?

Note the system is linear so superposition applies –

Consider superposition of oppositely rotating solutions –

$$\underset{\substack{\uparrow \\ \text{rotates}}}{e^{i\omega_1 t}} + \underset{\substack{\uparrow \\ \text{rotate}}}{e^{-i\omega_2 t}} = e^{i\left(\frac{\omega_1-\omega_2}{2}\right)t}\left[e^{i\left(\frac{\omega_1+\omega_2}{2}\right)t} + e^{-i\left(\frac{\omega_1+\omega_2}{2}\right)t}\right]$$

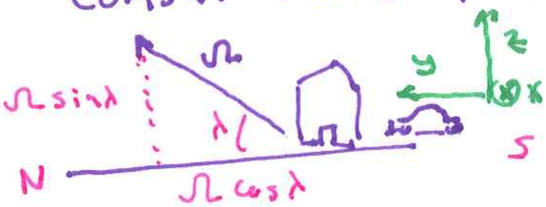
$$= e^{i\left(\frac{\omega_1-\omega_2}{2}\right)t}\left[\cos(\omega_0 t)\, 2\right]$$

$$\underset{\substack{\uparrow \\ \text{slow rotation}}}{e^{-i\Omega t}}\; 2\cos(\omega_0 t)$$

normal back & forth

$$T = \frac{T_0}{\sin\lambda} = \frac{24^h}{.707} \approx 34^h$$

Coriolis Force & vertical motion.



Note: for vertical motion (in contrast to horizontal motion) the vertical component of Ω is less important as $\overrightarrow{\Omega \times v}$ of that component $= 0$

$\vec{\Omega} = (0, \Omega\cos\lambda, \Omega\sin\lambda)$

Eq of motion: $\vec{a} = -2\overrightarrow{\Omega \times v} + \vec{g}$ ← centrifugal force included in this g

pure drop: 0^{th} approx: $\vec{r} = (0, 0, -\frac{1}{2}gt^2)$

$\vec{v} = (0, 0, -gt)$

$$\overrightarrow{\Omega \times v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \Omega\cos\lambda & \Omega\sin\lambda \\ 0 & 0 & -gt \end{vmatrix} = (\underbrace{-\Omega\cos\lambda\, g t}, 0, 0)$$

Note: only $\cos\lambda$ in this first effect

$$\vec{a} = (2\Omega\cos\lambda\, g t, 0, -g)$$
$$\vec{v} = (\Omega\cos\lambda\, g t^2, 0, -gt)$$
$$\vec{r} = (\underbrace{\tfrac{1}{3}\Omega\cos\lambda\, g t^3}, 0, -\tfrac{1}{2}gt^2)$$

deflection to East; $t = \sqrt{\frac{2h}{g}}$ → 1.5 cm

$h = 100$ m
$\lambda = 45^\circ$
$\Omega = \frac{2\pi}{24 \cdot 3600}$

1st Approx

$$\overrightarrow{\Omega \times v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \Omega\cos\lambda & \Omega\sin\lambda \\ \Omega\cos\lambda\, g t^2, & 0 & -gt \end{vmatrix} = (-\Omega\cos\lambda\, g t, \overbrace{\Omega^2\cos\lambda\sin\lambda\, g t^2}^{\text{small but compared to zero, important}}, \underbrace{-\Omega^2\cos^2\lambda\, g t^2}_{})$$

This is so small compared to g that it is not worth retaining.

$$\vec{a} = (2\Omega\cos\lambda\, g t, -2\Omega^2\sin\lambda\cos\lambda\, g t^2, -g)$$
$$\vec{v} = (\Omega\cos\lambda\, g t^2, -\tfrac{2}{3}\Omega^2\sin\lambda\cos\lambda\, g t^3, -gt)$$
$$\vec{r} = (\tfrac{1}{3}\Omega\cos\lambda\, g t^2, \underbrace{-\tfrac{1}{6}\Omega^2\sin\lambda\cos\lambda\, g t^4}_{1.8\,\mu m}, -\tfrac{1}{2}gt^2)$$

Note: we have neglected equally insignificant things like variation of $\vec{g}$ with altitude (centrifugal & gravity both vary with altitude)

Up & Down: $\vec{v} = (0, 0, v_0 - gt)$

$$\overrightarrow{\Omega \times v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \Omega\cos\lambda & \Omega\sin\lambda \\ 0 & 0 & v_0 - gt \end{vmatrix} \Rightarrow (\Omega\cos\lambda(v_0 - gt), 0, 0)$$

initial value

$$\vec{a} = (-2\Omega\cos\lambda(v_0 - gt), 0, -g)$$

$$\vec{v} = (-2\Omega\cos\lambda(v_0 t - \tfrac{1}{2}gt^2), 0, v_0 - gt)$$

$$\vec{r} = (\underbrace{-2\Omega\cos\lambda\left(\frac{v_0 t^2}{2} - \frac{1}{6}gt^3\right)}_{\substack{\text{for small } t \text{ this is } (-) \\ \text{but for larger } t \text{ might be } (+)}}, 0, v_0 t - \tfrac{1}{2}gt^2)$$

←west

←East

$t = \dfrac{2v_0}{g}$: $-\Omega\cos\lambda\, t^2\left(v_0 - \frac{1}{3}gt\right) = -\frac{1}{3}\Omega\cos\lambda\left(\frac{2v_0}{g}\right)^2 v_0$

$v_0 = 45\,\text{m/s} \longrightarrow .7\,\text{cm}$