Main result from normal modes: If there exist an equilibrium point ($\frac{\partial U}{\partial q_i}=0 \ \forall i$) then for small deviations from that equilibrium we can Taylor expand U thru quadratic terms ($U \approx U_0 + \frac{1}{2}(q)^T K (q)$)

K_{ij} matrix $= \frac{\partial^2 U}{\partial q_i \partial q_j}$
evaluate derivatives at equilibrium point

Lagrange $\Rightarrow$ $\underbrace{M(\ddot{q}) = -K(q)}$

mass matrix
$M_{ij} = \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}$

try solution $(q) = \vec{a}\, e^{i\omega t}$
constant vector
(of course we mean to take Re part)

$(\underbrace{K - \omega^2 M})\vec{a} = 0$

non trivial solution requires
det $(K - \omega^2 M) = 0$
$\Rightarrow$ normal mode "eigenfrequencies" ω_i
and corresponding "eigenvectors" $\vec{a}_i$

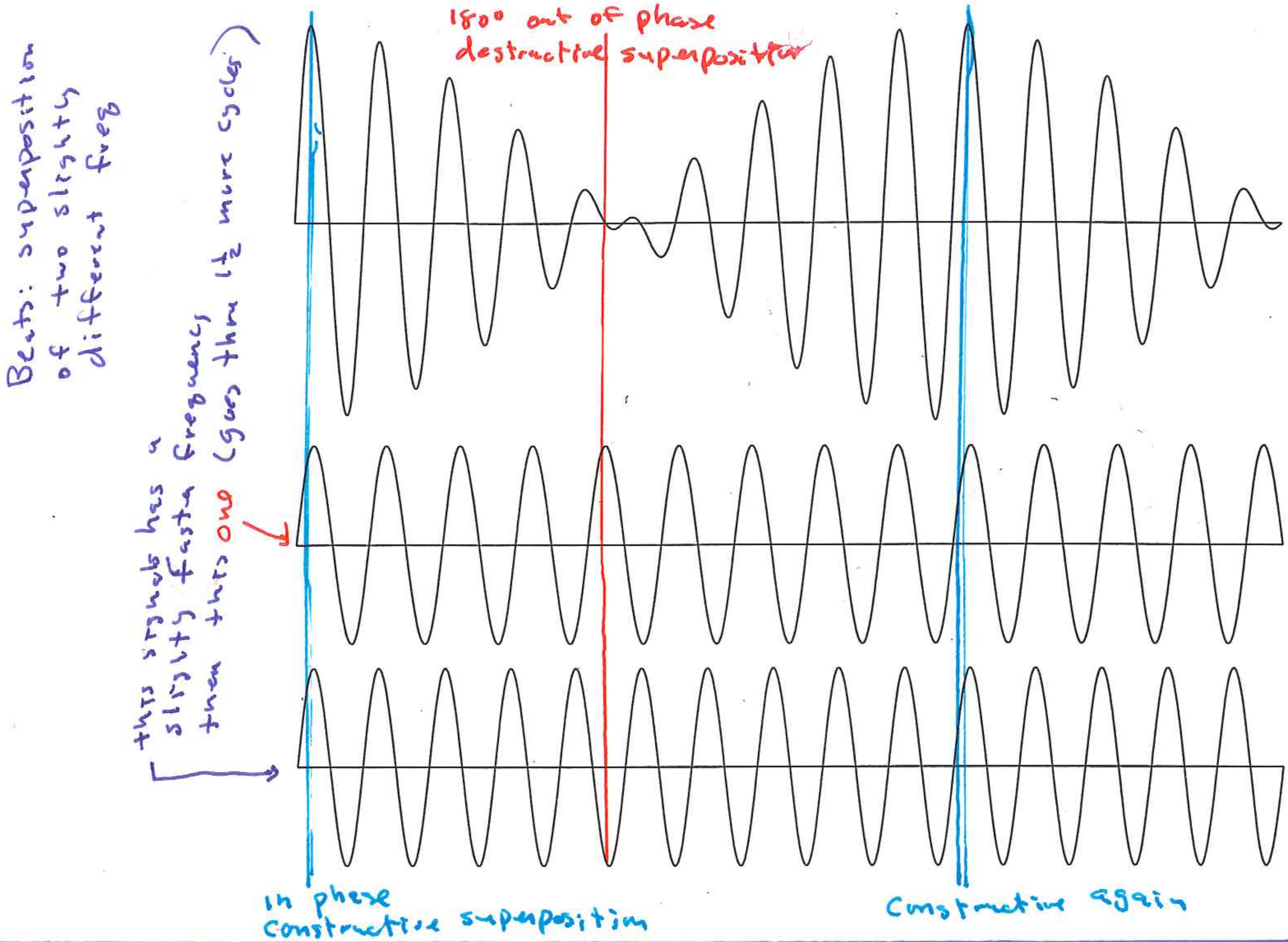
Switch to "normal mode coordinates" $Q_i \Rightarrow$

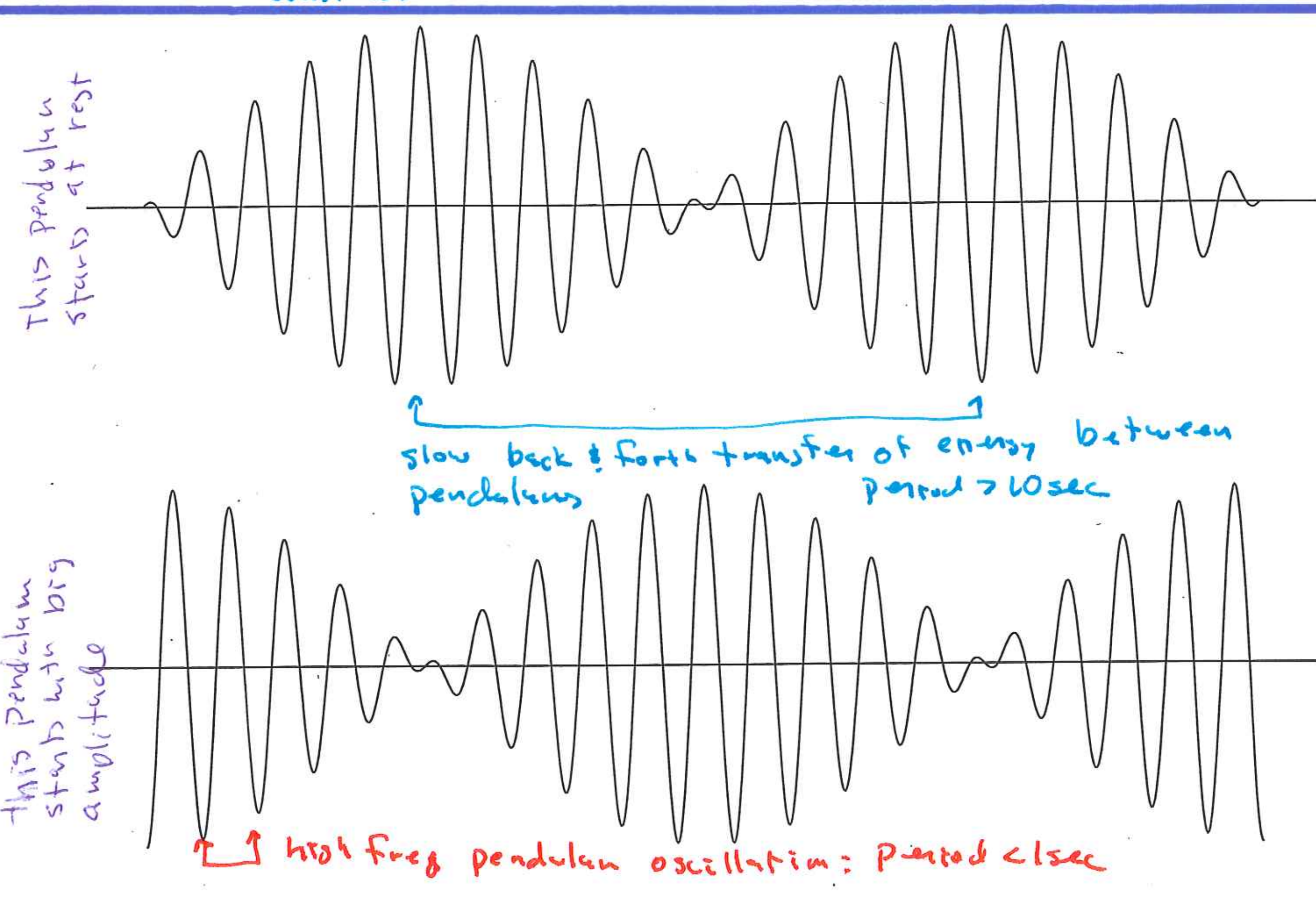
$L = \sum \frac{1}{2}\dot{Q}_i^2 - \frac{1}{2}\omega_i^2 Q_i^2 \leftarrow$ ie sum of non interacting SHO

Note: while the solution is the sum of SHO the solution may not <u>look</u> like such a sum.

Demo: Two identical pendulums are supported on a single

horizontal string — as one pendulum moves the support string moves a bit which jostles the other pendulum. The resulting motion (graphed bottom following page): Starting just one pendulum gets the other moving a bit. Over time the second pendulum grabs the energy of the first. After several seconds the first pendulum is at rest and the second pendulum has grabbed 100% of energy. The energy continues to cycle back & forth between the two pendulums.


Beats: superposition of two slightly different freq
180° out of phase destructive superposition
this signal has a slightly faster frequency then this one (goes thru 1/2 more cycles)
in phase constructive superposition
Constructive again


This pendulum starts at rest
slow back & forth transfer of energy between pendulums period > 10 sec
This pendulum starts with big amplitude
high freq pendulum oscillation: period < 1 sec

The two coupled pendulum demo is related to "Beats" which is described top of previous page. Two signals of slightly different freq are superimposed – when in phase there is a big resulting sum; when 180° out of phase the sum is zero. Because the two have slightly different frequencies they consistently switch between in-phase & out-of-phase.

Mathematics of Coupled Pendulum Demo ... you worked coupled pendulums as a homework problem & found symmetric & antisymmetric normal modes at slightly different freq $\rightarrow$ $\begin{pmatrix}1\\1\end{pmatrix}e^{i\omega_1 t}$ & $\begin{pmatrix}1\\-1\end{pmatrix}e^{i\omega_2 t}$

Consider a superposition of an equal mix of both –

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}e^{i\omega_1 t} + \begin{pmatrix}1\\-1\end{pmatrix}e^{i\omega_2 t} = \begin{pmatrix}e^{i\omega_1 t}+e^{i\omega_2 t}\\ e^{i\omega_1 t}-e^{i\omega_2 t}\end{pmatrix}$$

Consider top term first: $e^{i\omega_1 t}+e^{i\omega_2 t} = e^{i\left(\frac{\omega_1+\omega_2}{2}\right)t}\left[e^{i\left(\frac{\omega_1-\omega_2}{2}\right)t}+e^{-i\left(\frac{\omega_1-\omega_2}{2}\right)t}\right]$

$$= e^{i\left(\frac{\omega_1+\omega_2}{2}\right)t}\,2\cos\left(\frac{\omega_1-\omega_2}{2}t\right)$$

when take Re[] $= 2\cos\left(\underbrace{\frac{\omega_1+\omega_2}{2}}_{\text{avg freq}}t\right)\cos\left(\underbrace{\frac{\omega_1-\omega_2}{2}}_{\text{difference freq (much smaller)}}t\right)$

$$e^{i\omega_1 t}-e^{i\omega_2 t} = e^{i\left(\frac{\omega_1+\omega_2}{2}\right)t}\left[e^{i\left(\frac{\omega_1-\omega_2}{2}\right)t}-e^{-i\left(\frac{\omega_1-\omega_2}{2}\right)t}\right]$$

$$= e^{i\left(\frac{\omega_1+\omega_2}{2}\right)t}\,2i\sin\left(\frac{\omega_1-\omega_2}{2}t\right)$$

when take Re[] $= -2\sin\left(\frac{\omega_1+\omega_2}{2}t\right)\sin\left(\frac{\omega_1-\omega_2}{2}t\right)$

Result: both x_1 & x_2 seen to oscillate at the avg freq but the amplitude of that oscillations varies as $\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{\omega_1-\omega_2}{2}t\right)$ ie a slowly changing amplitude; since $\cos^2\theta+\sin^2\theta=1$ if one pendulum has a big amplitude the other amplitude is small

Remark: looking at the two pendulums you see a fast pendulum freq ($\sim \sqrt{g/L}$) and a slow back & forth transfer of energy. However any mathematically correct way of extracting freq (eg FFT - fast fourier transform) will show the situation correctly: two closely spaced frequencies.

Remark: It is often said that Laplace "proved that the Solar System is stable" which is remarkable because in fact it is not stable. While I've never looked up Laplace's writings myself, I assume Laplace was proceeding with the sort of Taylor Series expansion that we've done here with normal modes. Poincare is often credited with being the first to recognise that the Solar System could be chaotic. Chaos is the topic of the next chapter of our textbook (which we are not going to cover)

I'd like to add a few words about the observational data RE solar system. Observations over a century show many parameters of planetary orbit seem <u>periodic</u> as the normal modes method suggests. However some parameters show linear change (called secular change) so the question begins: is this apparent linear change over a century just a long period cycle?

Wiki reports a study integrating the equations of motion forward 5×10^9 years — in about 1% of the test cases Mercury's orbit had been perturbed enough to have it run into Venus!

Example 2 of unusual / unexpected normal modes. Foucault pendulum.

From exam 3: $\ddot{x} + \omega_0^2 x - 2\Omega \dot{y} = 0$

$\ddot{y} + \omega_0^2 y + 2\Omega \dot{x} = 0$

Try normal mode: $\begin{pmatrix} x \\ y \end{pmatrix} = \vec{q}\, e^{i\omega t}$: $-\omega^2 q_1 + \omega_0^2 q_1 - 2i\Omega\omega q_2 = 0$

$-\omega^2 q_2 + \omega_0^2 q_2 + 2i\Omega\omega q_1 = 0$

$$\underbrace{\begin{pmatrix} \omega_0^2 - \omega^2 & -2i\Omega\omega \\ 2i\Omega\omega & \omega_0^2 - \omega^2 \end{pmatrix}}\vec{q} = 0 \quad \text{matrix} \longleftarrow \begin{cases} (\omega_0^2 - \omega^2) q_1 - 2i\Omega\omega q_2 = 0 \\ 2i\Omega\omega\, q_1 + (\omega_0^2 - \omega^2) q_2 = 0 \end{cases}$$

nontrivial solution requires this to have det = 0

but first let's make an approx: $(\omega_0^2 - \omega^2) = \underbrace{(\omega_0 + \omega)}_{\approx 2\omega}(\omega_0 - \omega)$

$$\Rightarrow 2\omega \underbrace{\begin{pmatrix} \omega_0 - \omega & -i\Omega \\ i\Omega & \omega_0 - \omega \end{pmatrix}}\vec{q} = 0$$

$\det = 0 \Rightarrow (\omega_0 - \omega)^2 - \Omega^2 = 0$

$\omega_0 - \omega = \pm\Omega$

$\omega_0 \mp \Omega = \omega$

seek corresponding $\vec{q}$ ... start with $\omega = \omega_0 + \Omega$

$\Rightarrow \begin{pmatrix} -\Omega & -i\Omega \\ i\Omega & -\Omega \end{pmatrix}\vec{q} = \Omega \begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix}\vec{q} = 0$ so $-q_1 - iq_2 = 0$

$q_2 = iq_1$

$\Rightarrow \begin{pmatrix} 1 \\ i \end{pmatrix}$

seek corresponding $\vec{q}$ for $\omega = \omega_0 - \Omega$

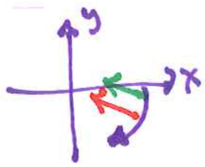
$\begin{pmatrix} \Omega & -i\Omega \\ i\Omega & \Omega \end{pmatrix}\vec{q} = \Omega \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\vec{q} = 0$ so $q_1 - iq_2 = 0$

$q_2 = -iq_1$

$\Rightarrow \begin{pmatrix} 1 \\ -i \end{pmatrix}$

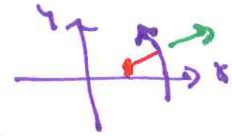
physics of these results: if $\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\begin{pmatrix} 1 \\ i \end{pmatrix} e^{i\omega_1 t}$

$= \begin{pmatrix} \cos\omega_1 t \\ -\sin\omega_1 t \end{pmatrix}$



coriolis force
pendulum force

both point in same direction
larger centripetal force ⇒ large frequency .. $\omega = \omega_0 + \Omega$

if $\begin{pmatrix}x\\y\end{pmatrix} = \mathrm{Re}\begin{pmatrix}1\\-i\end{pmatrix}e^{i\omega_2 t} = \begin{pmatrix}\cos\omega_2 t\\ \sin\omega_2 t\end{pmatrix} \Rightarrow$ 

opposite directions so smaller centripetal force & reduced freq: $\omega = \omega_0 - \Omega$ $\leftarrow$ (Pendulum force / Coriolis force)

Seek superposition of equal amounts of these two solutions

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\i\end{pmatrix}e^{i(\omega_0+\Omega)t} + \begin{pmatrix}1\\-i\end{pmatrix}e^{i(\omega_0-\Omega)t}$$

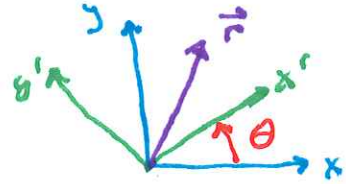
$$= \begin{pmatrix}e^{i\Omega t}+e^{-i\Omega t}\\ i(e^{i\Omega t}-e^{-i\Omega t})\end{pmatrix}e^{i\omega_0 t} = \begin{pmatrix}2\cos\Omega t\\ -2\sin\Omega t\end{pmatrix}e^{i\omega_0 t}$$

$$= 2\begin{pmatrix}\cos\omega_0 t\,\cos\Omega t\\ -\cos\omega_0 t\,\sin\Omega t\end{pmatrix} = \underbrace{2\cos\omega_0 t}_{\text{fast back \& forth motion of pendulum}}\underbrace{\begin{pmatrix}\cos\Omega t\\ -\sin\Omega t\end{pmatrix}}_{\text{slowly varying vector: rotates clockwise}}$$

Note: the slow rotation of pendulums oscillation direction (approx 40 hours) combined with few second basic pendulum back & forth is actually the result of 2 normal modes of nearly the same frequency what is a vector - A vector is a 3 tuple that when you change coordinate system changes exactly like $\vec{r}$

Remark: we showed in 2d:

$$\vec{r}\,' = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \vec{r}$$



we then put together 3 such 2d rotations (φ, θ, ψ) in directions (z, x', z'') to make a general 3d rotation.

we start then with just one vector $(\vec{r})$ and seek to make addition vectors.

Note: we require all physics eqs to be vector eqs so if they are true in one coordinate system they are true in all rotated coordinate systems as

$$\vec{A} = \vec{B} \Rightarrow \mathcal{M}\vec{A} = \mathcal{M}\vec{B} \Rightarrow \vec{A}\,' = \vec{B}\,'$$

the rotation matrix

what is a scalar - a quantity that does NOT change if coordinate system is rotated - example: mass change time

we can make new vectors using differences as

$$\Delta\vec{B} = \overrightarrow{B_1 - B_2} \Rightarrow \mathcal{M}\,\Delta\vec{B} = \mathcal{M}(\overrightarrow{B_1 - B_2}) = \mathcal{M}\vec{B}_1 - \mathcal{M}\vec{B}_2 = \Delta\vec{B}\,'$$

we can make new vectors by scaler multiplication

$$\vec{F} = m\vec{a} \Rightarrow \mathcal{M}\vec{F} = m\,\mathcal{M}\vec{a} \Rightarrow \vec{F}\,' = m\vec{a}\,'$$

Note: the force of gravity & Coulombs Law force depend on $\vec{r}$

$$F = \underbrace{Gm_1m_2 \frac{\vec{r}}{r^3}}_{} \quad \text{or} \quad Kq_1q_2 \frac{\vec{r}}{r^3}$$ and hence are vectors

see that this is a $\frac{1}{r^2}$ force!

If we add up these force vectors from lots of source particles the result is clearly still a vector