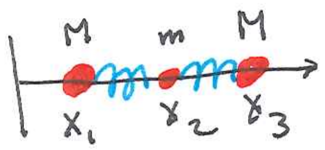

$$T = \frac{1}{2}\left(M\dot{x}_1^2 + m\dot{x}_2^2 + M\dot{x}_3^2\right) = \frac{1}{2}\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix}^T \begin{pmatrix}M & 0 & 0\\ 0 & m & 0\\ 0 & & M\end{pmatrix}\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix}$$

$$U = \frac{1}{2}k\left[(x_2 - x_1 - \ell)^2 + (x_3 - x_2 - \ell)^2\right]$$

Note: U has a min (hence zero force) @ $x_1 = K$, $x_2 = K+\ell$, $x_3 = K+2\ell$

seek behavior near that min; hence $x_1 = K + x_1'$ etc

$$U(x_1', x_2', x_3') = \frac{1}{2}\begin{pmatrix}x_1'\\ x_2'\\ x_3'\end{pmatrix}^T \Big[\ U\ \Big]\begin{pmatrix}x_1'\\ x_2'\\ x_3'\end{pmatrix} \qquad [U]_{ij} = \frac{\partial^2 U}{\partial x_i \partial x_j}$$

$$\frac{\partial^2 U}{\partial x_1^2} = k \qquad \frac{\partial^2 U}{\partial x_2^2} = 2k \qquad \frac{\partial^2 U}{\partial x_3^2} = k \qquad \frac{\partial^2 U}{\partial x_1 \partial x_2} = -k \qquad \frac{\partial^2 U}{\partial x_2 \partial x_3} = -k$$

Let $\vec{r} = \begin{pmatrix}x_1'\\ x_2'\\ x_3'\end{pmatrix} \Rightarrow L = \frac{1}{2}\dot{\vec{r}}^T\begin{pmatrix}M & 0 & 0\\ 0 & m & 0\\ 0 & 0 & M\end{pmatrix}\dot{\vec{r}} - \frac{1}{2}\vec{r}\begin{pmatrix}k & -k & 0\\ -k & 2k & 0\\ 0 & -k & k\end{pmatrix}\vec{r}$

Note: if $Q = \frac{1}{2}\sum_{ij} q_i A_{ij} q_j$ then $\frac{\partial Q}{\partial q_k} = \frac{1}{2}\left(\sum A_{kj} q_j + \sum q_i Q_{ik}\right)$

$$= \sum A_{kj} q_j$$

$$\Rightarrow \frac{\partial Q}{\partial \vec{q}} = \Big[A\Big]\Big[q\Big]$$

Lagrange Eqs: $\begin{pmatrix}M & 0 & 0\\ 0 & m & 0\\ 0 & 0 & M\end{pmatrix}\ddot{\vec{r}} = -\begin{pmatrix}k & -k & 0\\ -k & 2k & \cancel{k}\\ 0 & -k & k\end{pmatrix}\vec{r}$

seek solution $\vec{r} = \underset{\substack{\uparrow\\ \text{constant}}}{\vec{a}}\, e^{i\omega t} \Rightarrow \ddot{\vec{r}} = -\omega^2 \vec{a}\, e^{i\omega t}$

det( ) = 0 or $\vec{a} = \vec{0}$

$$\Rightarrow \begin{pmatrix}k - M\omega^2 & -k & 0\\ -k & 2k - m\omega^2 & -k\\ 0 & -k & k - M\omega^2\end{pmatrix}\vec{a} = \vec{0}$$

Note: if pull a constant from row/colum inside det( ) = k det( )

where $\omega_1^2 = \frac{k}{M}$ $\omega_0^2 = \frac{k}{m}$

$$\underset{\substack{\uparrow\\ \text{drop}}}{k^3}\det\begin{pmatrix}1 - \frac{\omega^2}{\omega_1^2} & -1 & 0\\ -1 & 2 - \frac{\omega^2}{\omega_0^2} & -1\\ 0 & -1 & 1 - \frac{\omega^2}{\omega_1^2}\end{pmatrix} = \left(1 - \frac{\omega^2}{\omega_1^2}\right)^2\left(2 - \frac{\omega^2}{\omega_0^2}\right) - 2\left(1 - \frac{\omega^2}{\omega_1^2}\right)$$

$$= \left(1 - \frac{\omega^2}{\omega_1^2}\right)\left[\left(1 - \frac{\omega^2}{\omega_1^2}\right)\left(2 - \frac{\omega^2}{\omega_0^2}\right) - 2\right]$$

$$= \left(1 - \frac{\omega^2}{\omega_1^2}\right)\left[\underbrace{\frac{\omega^2}{\omega_1^2\omega_0^2}}_{\omega = 0}\left(\omega^2 - \left(\omega_1^2 + 2\omega_0^2\right)\right)\right]$$

$\omega^2 = \omega_1^2$

$$\begin{pmatrix}0 & -1 & 0\\ -1 & X & -1\\ 0 & -1 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix} = 0$$

$$\begin{pmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\\ 1\end{pmatrix} = 0$$

$$\frac{\omega^2}{\omega_1^2} = 1 + 2\frac{\omega_0^2}{\omega_1^2} = 1 + 2\frac{M}{m}$$

$$\frac{\omega^2}{\omega_0^2} = \frac{\omega_1^2}{\omega_0^2} + 2 = 2 + \frac{m}{M}$$

$$\begin{pmatrix}-2\frac{M}{m} & -1 & 0\\ -1 & -\frac{m}{M} & -1\\ 0 & -1 & -2\frac{M}{m}\end{pmatrix}\begin{pmatrix}1\\ -2M/m\\ 1\end{pmatrix} = 0$$

Change of variables: $\vec{r} = q_1 \begin{pmatrix}1\\1\\1\end{pmatrix} + q_2 \begin{pmatrix}1\\0\\-1\end{pmatrix} + q_3 \begin{pmatrix}1\\-2M/m\\1\end{pmatrix}$

$$\begin{pmatrix} M & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & M \end{pmatrix} \left\{ \dot{q}_1 \begin{pmatrix}1\\1\\1\end{pmatrix} + \dot{q}_2 \begin{pmatrix}1\\0\\-1\end{pmatrix} + \dot{q}_3 \begin{pmatrix}1\\-2M/m\\1\end{pmatrix} \right\} = \dot{q}_1 \begin{pmatrix}M\\m\\M\end{pmatrix} + \dot{q}_2 \begin{pmatrix}M\\0\\-M\end{pmatrix} + \dot{q}_3 \begin{pmatrix}M\\-2M\\M\end{pmatrix}$$

Now dot on left $\left\{ \dot{q}_1 \begin{pmatrix}1\\1\\1\end{pmatrix} + \dot{q}_2 \begin{pmatrix}1\\0\\-1\end{pmatrix} + \dot{q}_3 \begin{pmatrix}1\\-2M/m\\1\end{pmatrix} \right\}^T$

connected terms are zero → diagonals only

$$T = \frac{1}{2} \left\{ \dot{q}_1^2 (2M+m) + \dot{q}_2^2 (2M) + \dot{q}_3^2 \left(1 + 2\frac{M}{m}\right) 2M \right\}$$

$$\begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} \left\{ q_1 \begin{pmatrix}1\\1\\1\end{pmatrix} + q_2 \begin{pmatrix}1\\0\\-1\end{pmatrix} + q_3 \begin{pmatrix}1\\-2M/m\\1\end{pmatrix} \right\} = q_2 \begin{pmatrix}k\\0\\-k\end{pmatrix} + q_3 \begin{pmatrix}1+\frac{2M}{m}\\-2\left(1+\frac{2M}{m}\right)\\1+\frac{2M}{m}\end{pmatrix} k$$

Now dot on left $\left\{ q_1 \begin{pmatrix}1\\1\\1\end{pmatrix} + q_2 \begin{pmatrix}1\\0\\-1\end{pmatrix} + q_3 \begin{pmatrix}1\\-2M/m\\1\end{pmatrix} \right\}$

connected terms are zero → diagonals only

$$U = \frac{1}{2} \left\{ q_2^2\, 2k + q_3^2 \left[2\left(1+\frac{2M}{m}\right) + 2\frac{M}{m}\, 2\left(1+\frac{2M}{m}\right) \right] k \right\}$$

$$= 2\left(1+\frac{2M}{m}\right)\left[1+\frac{2M}{m}\right] = 2\left(1+\frac{2M}{m}\right)^2$$

$$L = \underbrace{\frac{1}{2}(2M+m)\dot{q}_1^2}_{\dot{q}_1 = \text{const}} + \underbrace{\frac{1}{2} 2M \dot{q}_2^2 - \frac{1}{2} 2k q_2^2}_{\text{"}\omega^2\text{"} = \frac{2k}{2M} = \frac{k}{M}} + \frac{1}{2}\left(1+\frac{2M}{m}\right) 2M \dot{q}_3^2 - \frac{1}{2} 2\left(1+\frac{2M}{m}\right)^2 k q_3^2$$

SHO

$$\text{"}\omega^2\text{"} = \frac{2k\left(1+\frac{2M}{m}\right)^2}{2M\left(1+\frac{2M}{m}\right)} = \frac{k}{M}\left(1+\frac{2M}{m}\right)$$

$$= \omega_1^2 + 2\omega_2^2$$

Result: motion is sum of 3 unrelated motions:

Constant velocity → q_1

SHO @ $\sqrt{\frac{k}{m}}$ → q_2

SHO @ $\sqrt{k\left(\frac{1}{M}+\frac{2}{m}\right)}$ → q_3

IF desired can go back to $\begin{pmatrix}x_1'\\x_2'\\x_3'\end{pmatrix} = \vec{r} = q_1 \begin{pmatrix}1\\1\\1\end{pmatrix} + q_2 \begin{pmatrix}1\\0\\-1\end{pmatrix} + q_3 \begin{pmatrix}1\\-2M/m\\1\end{pmatrix}$