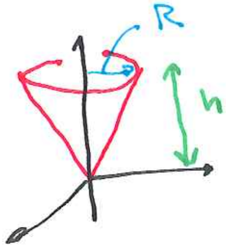
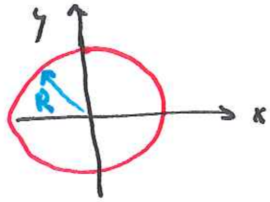
More complex $\overleftrightarrow{I}$ problem — ice cream cone from vertex




Begin with simpler problem of ring then use parallel axis thm for elevated ring. Then assemble rings to make cone.




All the mass is at $z=0$ so $I_{xz} = I_{yz} = 0$

$I_{zz}$ is easy as $\forall$ mass $x^2+y^2 = R^2$

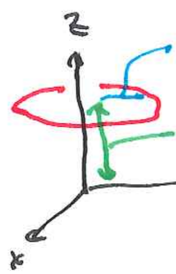
$\longrightarrow I_{zz} = MR^2$

$I_{xy} = 0$ as there is a symmetry plane or as in disk: $\int \sin\theta \cos\theta \, d\theta = 0$

As with any lamina $I_{xx} + I_{yy} = I_{zz}$ and $I_{xx} = I_{yy}$ by symmetry so: $\overleftrightarrow{I} = MR^2 \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Parallel axis Thm: $\underset{\substack{\uparrow \\ \text{uses new orgin}}}{\overleftrightarrow{J}} = \underset{\substack{\uparrow \\ \text{about CM}}}{\overleftrightarrow{I}} + M \left\{ a^2 \overleftrightarrow{1} - \vec{a}\vec{a} \right\}$ same terms as usual $\overleftrightarrow{I}$ just use $\vec{a}$

$\vec{a}$ connects CM with new orgin




$r = \frac{R}{h} z$

height $z$: $\vec{a} = (0, 0, z)$

$$\{\ \} = \begin{bmatrix} z^2 & 0 & 0 \\ 0 & z^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

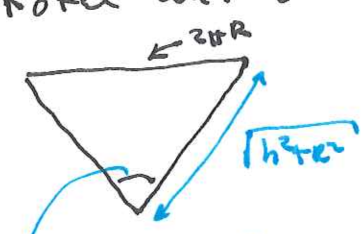
$$\overleftrightarrow{I} = M \begin{bmatrix} R^2/2 + z^2 & 0 & 0 \\ 0 & \frac{R^2}{2} + z^2 & 0 \\ 0 & 0 & R^2 \end{bmatrix}$$

As a piece of the cone the ring has radius $r = \frac{R}{h} z$

Area of a section of cone: $2\pi r \sqrt{\Delta z^2 + \Delta r^2} = 2\pi r \sqrt{1 + \left(\frac{R}{h}\right)^2} \Delta z$

$\leftarrow \sqrt{\Delta R^2 + \Delta z^2} = \sqrt{r'^2 + 1} \, \Delta z$

Total area of cone: $\int_0^h 2\pi \left(\frac{R}{h} z\right) \sqrt{1 + \left(\frac{R}{h}\right)^2} \, dz = \pi R \sqrt{h^2 + R^2}$




$\text{Area} = \frac{\theta}{2\pi} \pi \left(\sqrt{h^2 + R^2}\right)^2 = \pi R \sqrt{h^2 + R^2}$

same result either way

$\theta = \frac{2\pi R}{\sqrt{h^2 R^2}}$

Note: $I_{xx} = I_{yy} = \frac{M}{\text{Area}} \int \left(\frac{r^2}{2} + z^2\right) dm$

$r = \frac{R}{h} z \qquad dm = 2\pi r \sqrt{1 + \left(\frac{R}{h}\right)^2}\, dz$

$$= \frac{M}{\text{Area}}\, 2\pi \sqrt{1 + \left(\frac{R}{h}\right)^2} \int \left[\frac{1}{2}\left(\frac{R}{h}\right)^2 + 1\right] \underbrace{z^2\, \frac{R}{h}\, z \; dz}_{\frac{h^4}{4}}$$

$$= \frac{M}{\text{Area}}\, 2\pi \sqrt{h^2 + R^2} \left[\frac{1}{2} R^2 + h^2\right] R\, \frac{1}{4}$$

$\text{Area} = \pi R \sqrt{h^2 + R^2}$

$$= \frac{1}{2} M \left[\frac{1}{2} R^2 + h^2\right]$$

$dm = 2\pi r \sqrt{1 + \left(\frac{R}{h}\right)^2}\, dz \qquad dz = \frac{h}{R}\, dr$

$$I_{zz} = \frac{M}{\text{Area}} \int r^2\, dm$$

$$= \frac{M}{\text{Area}}\, 2\pi \sqrt{\left(\frac{h}{R}\right)^2 + 1} \int r^3\, dr$$

$\int r^3\, dr = \frac{1}{4} R^4 \qquad \text{Area} = \pi R \sqrt{h^2 + R^2}$

$$= \frac{1}{2} M R^2$$

Principal Axes — $\exists$ 3 mutually perpendicular axes

$\ni \quad \vec{L} = \overleftrightarrow{I} \cdot \vec{\omega} = \lambda \vec{\omega} \quad$ (ie $\vec{\omega} \parallel \vec{L}$)

{ eigenproblem of symmetric matrix $\overleftrightarrow{I}$ }

Principal Axes cube —

$$\vec{I} = \underbrace{\frac{Ma^2}{12}} \begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix} \longrightarrow \det \begin{bmatrix} 8-x & -3 & -3 \\ -3 & 8-x & -3 \\ -3 & -3 & 8-x \end{bmatrix} = 0$$

overall factor just affects overall value of eigenvalues not directions of eigenvectors

$$= (8-x)^3 - 2\cdot 3^3 - 3\cdot 3^2(8-x)$$

$$-x^3 + 3\cdot 8x^2 - 3\cdot 8^2 x + 8^3$$

$$+27x$$

$$-x^3 + 24x^2 - 165x$$

$$8^3 - 2\cdot 27 - 8\cdot 27 = 242$$

$$\Rightarrow \quad x^3 - 24x^2 + 165x - 242 = 0$$

If can find one root then just quadratic remains

Try eigenvector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad \begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \underset{\text{root}}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\begin{array}{r} x^2 - 22x + 121 \\ x-2 \,\big)\, x^3 - 24x^2 + 165x - 242 \\ -\quad x^3 - 2x^2 \qquad\qquad\qquad \\ \hline -22x^2 + 165x \qquad \\ -22x^2 + 44x \qquad \\ \hline 121x - 242 \end{array} \Bigg\} = (x-2)\underbrace{(x^2 - 22x + 121)}_{(x-11)^2}$$

Eigenvectors $\perp$ . i try $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 11 \\ -11 \\ 0 \end{pmatrix}$

Final $\perp$ to pair $\rightarrow$ cross product: $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = (-1, -1, 2)$

$$\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$