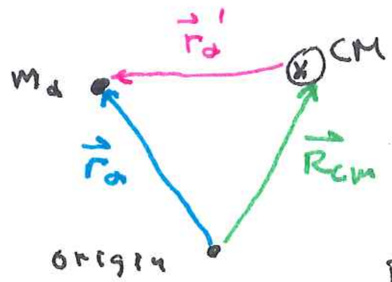
Chapter 10 starts with a review of Composite things.



$$\vec{R}_{cm} = \underbrace{\frac{1}{M}}_{\text{total mass}} \sum m_\alpha \vec{r}_\alpha$$

$$\vec{V}_{cm} = \frac{1}{M} \sum m_\alpha \vec{v}_\alpha$$

$$\vec{A}_{cm} = \frac{1}{M} \sum m_\alpha \vec{a}_\alpha$$

Useful Lemma:

$$\sum m_\alpha \vec{r}_\alpha\,' = 0$$

Because of Newton 3RD, internal forces cancel & $\vec{F}_{ext} = M\vec{A}$

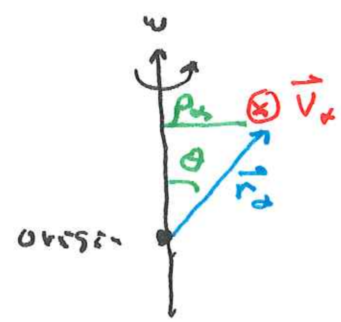
ie the source of the force is a particle in the system

$$\text{KE:} \quad T = \sum \tfrac{1}{2} m_\alpha v_\alpha^2 = \underbrace{\tfrac{1}{2} M V_{cm}^2}_{\text{"OF" CM}} + \underbrace{\sum \tfrac{1}{2} m_\alpha \vec{v}_\alpha\,'^2}_{\text{"ABOUT" CM}}$$

FYI: the textbook has not stressed that for uniform gravity the net force can be thought of as applied at CM

ie - $PE = \sum m_\alpha g z_\alpha = M g Z_{cm}$

torque: $\sum \vec{r}_\alpha \times m_\alpha \vec{g} = M \vec{R}_{cm} \times \vec{g}$



For rotating rigid bodies: $\vec{v}_\alpha = \vec{\omega} \times \vec{r}_\alpha$

$$|\vec{v}_\alpha| = |\vec{\omega}|\,|\vec{r}_\alpha| \sin\theta = \omega \rho_\alpha$$

$$\text{So } T = \sum \tfrac{1}{2} m_\alpha (\omega \rho_\alpha)^2 = \tfrac{1}{2} \underbrace{\sum m_\alpha \rho_\alpha^2}_{} \omega^2$$

we called this moment of inertia I but its about to become more complex

$$\text{Angular Momentum: } \vec{L} = \sum m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha$$

$$= \underbrace{M \vec{R}_{cm} \times \vec{V}_{cm}}_{\substack{\text{"OF" CM} \\ \text{orbital} \\ \text{depends on} \\ \text{origin}}} + \underbrace{\sum m_\alpha \vec{r}_\alpha\,' \times \vec{v}_\alpha\,'}_{\substack{\text{"ABOUT" CM} \\ \text{spin} \\ \text{does not depend} \\ \text{on origin}}}$$

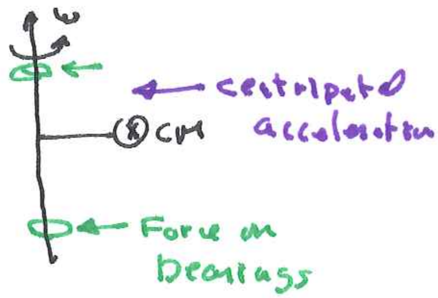
$$\frac{d\vec{L}_{orbit}}{dt} = \underbrace{\vec{R}_{cm} \times \vec{F}_{ext}}_{\substack{\text{Torque OF} \\ \text{CM}}}$$

$$\frac{d\vec{L}_{spin}}{dt} = \underbrace{\sum \vec{r}_\alpha\,' \times \vec{F}_{ext}}_{\substack{\text{Torque About} \\ \text{CM}}}$$

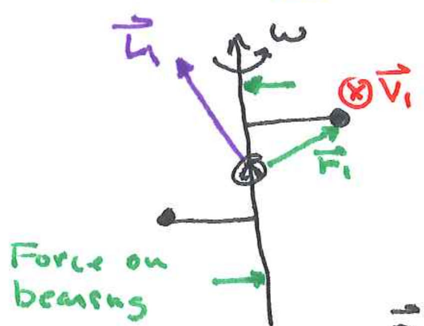
Note by above comments gravity ($m\vec{g}$) never has a torque about the CM

Balanced & not rotation

Case of CM off axis:



CM on axis but masses "unbalanced"



For other mass: $\vec{r}_2 = -\vec{r}_1$, $\vec{V}_2 = -\vec{V}_1$

so $\vec{L}_2 = \vec{L}_1$

so $\vec{L}_{total}$ must rotated

$\Rightarrow \frac{d\vec{L}}{dt} \neq 0 \Rightarrow$ Torque required

$\dot{\vec{L}}$ out-of-page

So in general $\vec{L}$ & $\vec{\omega}$ are not pointed in same direction so $\vec{L} = I\vec{\omega}$ not generally possible.

$\vec{\omega} \times \vec{r}_\alpha$

$$\vec{L} = \sum m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha = \sum m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)$$

$$= \sum m_\alpha \left\{ \vec{\omega}\, r_\alpha^2 - \underbrace{\vec{r}_\alpha (\vec{\omega} \cdot \vec{r}_\alpha)} \right\}$$

This term $\Rightarrow \sum m_\alpha r_\alpha^2 \vec{\omega}$ so result is in $\vec{\omega}$ direction

This is "problem" term.

More generally: $\vec{A}(\vec{B} \cdot \vec{C})$

$$= \left(A_x (B_x C_x + B_y C_y + B_z C_z),\; A_y (\quad),\; A_z (\quad) \right)$$

write as matrix:

$$\underbrace{\begin{bmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{bmatrix}}_{\text{"Dyadic" } \vec{A}\vec{B}} \begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} = (\vec{A}\vec{B}) \cdot \vec{C}$$

No dot or cross here - just juxtaposed

here $\vec{A} = \vec{r}_\alpha$ & $\vec{B} = \vec{r}_\alpha$

$$\text{so} \quad \begin{bmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{bmatrix}$$

$$L = \left\{ \begin{bmatrix} \sum m_\alpha r_\alpha^2 & 0 & 0 \\ 0 & \sum m_\alpha r_\alpha^2 & 0 \\ 0 & 0 & \sum m_\alpha r_\alpha^2 \end{bmatrix} - \begin{bmatrix} \sum m_\alpha x_\alpha x_\alpha & \sum m_\alpha x_\alpha y_\alpha & \sum m_\alpha x_\alpha z_\alpha \\ \sum m_\alpha y_\alpha x_\alpha & \sum m_\alpha y_\alpha y_\alpha & \sum m_\alpha y_\alpha z_\alpha \\ \sum m_\alpha z_\alpha x_\alpha & \sum m_\alpha z_\alpha y_\alpha & \sum m_\alpha z_\alpha z_\alpha \end{bmatrix} \right\}$$

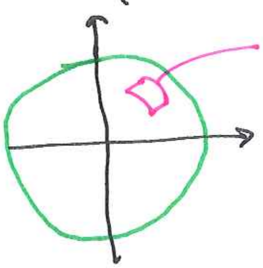
$$= \begin{bmatrix} \sum m_\alpha (r_\alpha^2 - x_\alpha^2) & -\sum m_\alpha x_\alpha y_\alpha & -\sum m_\alpha x_\alpha z_\alpha \\ -\sum m_\alpha y_\alpha x_\alpha & \sum m_\alpha (r_\alpha^2 - y_\alpha^2) & -\sum m_\alpha y_\alpha z_\alpha \\ -\sum m_\alpha z_\alpha x_\alpha & -\sum m_\alpha z_\alpha y_\alpha & \sum m_\alpha (\underbrace{r_\alpha^2 - z_\alpha^2}_{= x_\alpha^2 + y_\alpha^2}) \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

r is everything except z

$$= \overleftrightarrow{I} \cdot \vec{\omega}$$

moment of inertia tensor

Examples — Disk — "lamina" — thin sheet — all mass at $z=0$ plane



$dA = dr\, r\, d\phi \longrightarrow dm = \sigma\, dA$

$x = r\cos\phi$

$y = r\sin\phi$

$z = 0$

$$\frac{\text{mass}}{\text{Area}} = \sigma = \frac{M}{\pi R^2}$$

$$I_{xy} = -\int \sigma\, dr\, r\, d\phi\; xy = -\sigma \int r^2 \overbrace{\sin\phi\cos\phi}^{\to 0}\; r\, dr\, d\phi$$

$$I_{xx} = \int \sigma\, dr\, r\, d\phi\, (y^2 + \underset{\uparrow\, 0}{z^2}) = \sigma \int \underbrace{r^2}_{\to \frac{1}{4}R^4} \underbrace{\sin^2\phi}_{\frac{1}{2}\cdot 2\pi}\; r\, dr\, d\phi$$

$$= \frac{M}{\pi R^2}\, \frac{1}{4} R^4 \pi = \frac{1}{4} M R^2$$

Note: $I_{yy} \longrightarrow \int \cos^2\phi\, d\phi = \frac{1}{2}\cdot 2\pi$ so same result

$$I_{zz} = \int \sigma\, dr\, r\, d\phi\, (x^2 + y^2) = I_{xx} + I_{yy} = \frac{1}{2} M R^2$$

see that this is a general result for lamina

$$\overleftrightarrow{I} = \frac{1}{2} M R^2 \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Cube at CM:

take side as 2s

integrals: $\int_{-s}^{s}$

$$I_{xy} = -\int \rho \, dx\,dy\,dz\,(xy)$$

$$\int_{-s}^{s} y\,dy = \tfrac{1}{2} y^2 \Big|_{-s}^{s} = 0$$

$$I_{xx} = \int \rho \, dx\,dy\,dz\,(y^2+z^2)$$

z will work just like y

$$\int dx \int dz \int_{-s}^{s} y^2\,dy$$

$$\underset{2s}{\int dx} \quad \underset{2s}{\int dz} \quad \underset{\frac{2s^3}{3}}{\int_{-s}^{s} y^2\,dy}$$

$$\rho = \frac{M}{(2s)^3}$$

$$= \rho \;\; 2s \cdot 2s \cdot \tfrac{2}{3} s^3 \times 2$$

$$= \frac{2}{3} M s^2$$

$\overleftrightarrow{I} = \frac{2}{3} M s^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ so here $\vec{L}$ & $\vec{\omega}$ point same direction