Some proofs –

In the general context of calculus of variations we showed that if the integrand (then called F, now called L) did not depend on the variable of integration (then x now t) then the following was constant:

$$\sum \frac{\partial L}{\partial \dot{q}_\alpha} \dot{q}_\alpha - L$$

We now show that in the usual case this constant is energy.

Require "natural" [no t] dependence of $\vec{r}$ [cartesian] on q_α: $\quad \vec{r} = \vec{r}(q_\alpha)$

PE has no $\dot{q}_\alpha$ dependence. $\Rightarrow \dfrac{\partial L}{\partial \dot{q}_\alpha} = \dfrac{\partial T}{\partial \dot{q}_\alpha}$

$$\vec{r} = \vec{r}(q_\alpha) \Rightarrow \vec{v} = \sum_\alpha \frac{\partial \vec{r}}{\partial q_\alpha} \dot{q}_\alpha \Rightarrow T = \frac{1}{2} m \left(\sum_\beta \frac{\partial \vec{r}}{\partial q_\beta} \dot{q}_\beta \right) \cdot \left(\sum_\alpha \frac{\partial \vec{r}}{\partial q_\alpha} \dot{q}_\alpha \right)$$

$$\frac{\partial T}{\partial \dot{q}_1} = \sum_{\alpha\beta} A_{\beta\alpha} \left\{ \frac{\partial \dot{q}_\beta}{\partial \dot{q}_1} \dot{q}_\alpha + \dot{q}_\beta \frac{\partial \dot{q}_\alpha}{\partial \dot{q}_1} \right\} = \sum_{\alpha,\beta} \underbrace{\frac{1}{2} m \frac{\partial \vec{r}}{\partial q_\beta} \cdot \frac{\partial \vec{r}}{\partial q_\alpha}} \dot{q}_\beta \dot{q}_\alpha$$

zero unless $\beta = 1$ — zero unless $\alpha = 1$ — a symmetric matrix we call $A_{\beta\alpha}$ just depends on q_α Not $\dot{q}_\alpha$

$$= \sum_\alpha A_{1\alpha} \dot{q}_\alpha + \underbrace{\sum_\beta A_{\beta 1} \dot{q}_\beta}_{\text{call } \beta\ \alpha \text{, use symmetric}}$$

$$= 2 \sum_\alpha A_{1\alpha} \dot{q}_\alpha$$

Now $\displaystyle\sum_\beta \frac{\partial T}{\partial \dot{q}_\beta} \dot{q}_\beta = 2 \sum_\beta \sum_\alpha A_{\beta\alpha} \dot{q}_\alpha \dot{q}_\beta = 2T$

so $\displaystyle\sum_\beta \frac{\partial T}{\partial \dot{q}_\beta} \dot{q}_\beta - L = 2T - (T - U) = T + V = \text{energy}$

In the future we'll call this Hamiltonian

$\vec{R} = \vec{r} + \vec{\varepsilon}$

variation — called this δq before

actual path

new path slightly different from actual path

$$\delta L = \underbrace{\frac{\partial L}{\partial \dot{r}} \dot{\varepsilon}}_{\text{usual integration by parts trick}} + \frac{\partial L}{\partial r} \varepsilon = \left[- \frac{d}{dt} \underbrace{\frac{\partial L}{\partial \dot{r}}}_{-m\vec{a}} + \underbrace{\frac{\partial L}{\partial r}}_{-\vec{\nabla} U = \vec{F}_{ext}} \right] \vec{\varepsilon}$$

$-\vec{F}_{constraint}$

$$= - \left[\vec{F}_{constraint} \right] \cdot \vec{\varepsilon}$$

But for constrained paths $\vec{\varepsilon} \perp \vec{F}_{constraint}$

so $\delta L = 0$

Relation between "displacement" symmetry & conservation.

→ L has no t dependence ⇒ t displacement symmetry

⇒ Energy conservation

→ L has no q_α dependence ⇒ $\frac{\partial L}{\partial \dot{q}_\alpha} = \text{constant}$

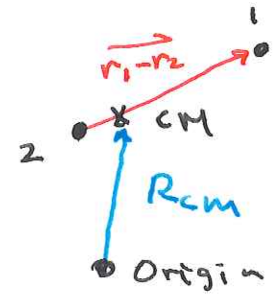
Define $\frac{\partial L}{\partial \dot{q}_\alpha}$ as "generalized / canonical momentum" ⇒ momentum conservation.

→ Note: if no external forces ⇒ U that depends on coordinate differences ⇒ if displace every coordinate the same L unchanged

$$0 = \underbrace{\frac{\partial L}{\partial \varepsilon}}_{\text{small displacement}} = \sum \frac{\partial L}{\partial q_\alpha} \underbrace{\frac{\partial q_\alpha}{\partial \varepsilon}}_{1} = \frac{d}{dt} \sum \frac{\partial L}{\partial \dot{q}_\alpha} = \frac{d}{dt} \underbrace{\sum p_\alpha}_{\text{total momentum conserved.}}$$

2 body systems: $\vec{r_1}$ & $\vec{r_2}$ (assume $m_2 > m_1$)
$\qquad\qquad\qquad\;\;$ m_1 $\quad$ m_2

$U(\vec{r_1}-\vec{r_2})$ → ie no external force - just depends on relative distance



$$T = \frac{1}{2} M V_{cm}^2 + \frac{1}{2}\mu v^2$$

$$\vec{V}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{M} \qquad v = \dot{\vec{r_1 - r_2}}$$

Since no external forces V_{cm} = constant

$$L = \frac{1}{2}\mu v^2 - U(\vec{r})$$

$$\text{angular momentum} = \vec{L} = \underbrace{M\,\vec{R_{cm}\times V_{cm}}}_{\text{orbital or "of CM"}} + \underbrace{\overbrace{\mu\vec{r}\times\vec{v}}^{\text{call }\vec{\ell}}}_{\text{spin or "about CM"}}$$

This will be constant as
$\vec{R}_{cm} = \vec{R}_{cm}(0) + \vec{V}_{cm}t$ ← $\vec{V}_{cm}\times\vec{V}_{cm} = 0$

For central forces this must be a constant as zero torque about CM

Remark: if $\vec{A}\times\vec{B} = \vec{C}$ then $\vec{A}$ & $\vec{B}$ must both be $\perp$ to $\vec{C}$ so $\vec{A}$ & $\vec{B}$ "live" in the plane $\perp$ to $\vec{C}$

Forces directed between 1 & 2 so $\vec{r\times F} = 0$

Here $\mu\vec{r}\times\vec{v} = \vec{\ell}$ = constant so $\vec{r}$ & $\vec{v}$ live in the plane $\perp$ to $\vec{\ell}$ → us polar coordinates

$$\left.\begin{aligned}\vec{r} &= r\hat{r}\\ \vec{v} &= \dot{r}\hat{r} + (r\dot{\phi})\hat{\phi}\end{aligned}\right\}\ \mu\,\vec{r\times v} = \mu\underbrace{(r^2\dot{\phi})}_{\text{must be constant} = \ell}\ \underbrace{\hat{r}\times\hat{\phi}}_{\text{call this constant vector }\hat{z}}$$

$$v^2 = (\dot{r}^2 + (r\dot{\phi})^2)$$

$$L = \frac{1}{2}\mu(\dot{r}^2 + (r\dot{\phi})^2) - U(r)$$

$$\boxed{r}\quad \mu\ddot{r} = \mu r\dot{\phi}^2 - \frac{\partial U}{\partial r} = \mu r\left(\frac{\ell}{\mu r^2}\right)^2 - \frac{\partial U}{\partial r} = \frac{\ell^2}{\mu r^3} - \frac{\partial U}{\partial r} = \frac{-\partial}{\partial r}\left(\frac{\ell^2}{2\mu r^2} + U\right)$$

$$\boxed{\phi}\quad \frac{d}{dt}\mu r^2\dot{\phi} = 0 \Rightarrow \mu r^2\dot{\phi} = \text{constant} = \ell$$