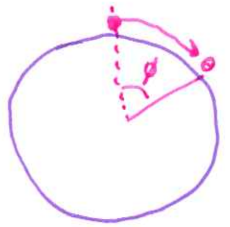
we showed previously that in rectangular coordinates

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q} = \frac{\partial L}{\partial q} \quad \text{was the same as} \quad m\ddot{\vec q} = -\vec\nabla U$$

but the real power of Lagrange comes with "generalized" coordinates — coordinates that suit the problem.

Today: multiple examples of Lagrange.



sliding down a radius R sphere

$$KE = \frac{1}{2} m R^2 \dot\phi^2$$

$$PE = mgR\cos\phi$$

$$L = \frac{1}{2} m R^2 \dot\phi^2 - mgR\cos\phi$$

$$\frac{\partial L}{\partial \dot\phi} = mR^2\dot\phi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = mR^2\ddot\phi$$

$$\frac{\partial L}{\partial \phi} = mgR\sin\phi$$

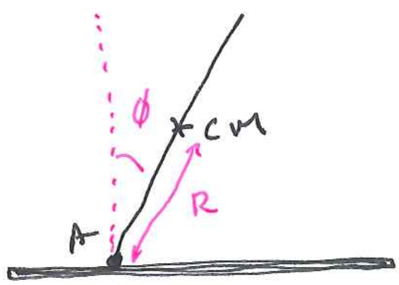
diff eq:

$$\ddot\phi = \frac{g}{R}\sin\phi$$

we showed in general Polar coordinates

$\vec v = \underbrace{\dot r\,\hat r}_{\text{radial velocity}} + \underbrace{r\dot\phi\,\hat\phi}_{\text{tangential velocity}}$

Here $r = R$ so $\dot r = 0$



Ladder falling (rotating) to ground

$$KE = \frac{1}{2} I_A \dot\phi^2$$

$$PE = mgR\cos\phi$$

$$L = \frac{1}{2} I_A \dot\phi^2 - mgR\cos\phi$$

$$\frac{\partial L}{\partial \dot\phi} = I_A\dot\phi$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\phi} = I_A\ddot\phi$$

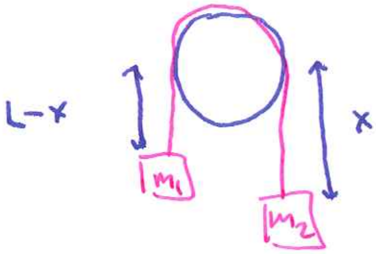
$$\frac{\partial L}{\partial \phi} = mgR\sin\phi$$

Remark: $KE = \frac{1}{2} I \omega^2$ where I is for origin on axis

Here $I_A = \frac{4}{3} mR^2 \qquad I_{CM} = \frac{1}{3} mR^2$

Parallel axis says: $I_A = I_{CM} + mR^2$ ✓

Atwoods Machine:



$KE = \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} m_1 \dot{x}^2$

(note: m_1 & m_2 are moving in different directions but KE only cares about speed)

$PE = m_2 g(-x) + m_1 g\left[-(L-x)\right]$

$= g(m_1 - m_2)x$ + constant

$L = \frac{1}{2}(m_1+m_2)\dot{x}^2 - g(m_1-m_2)x$ + constant

$\frac{\partial L}{\partial \dot{x}} = (m_1+m_2)\dot{x}$ $\qquad$ $\frac{\partial L}{\partial x} = -g(m_1-m_2)$

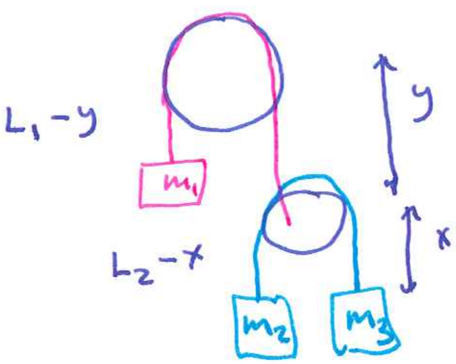
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (m_1+m_2)\ddot{x}$

$(m_1+m_2)\ddot{x} = -g(m_1-m_2)$

$\ddot{x} = g\frac{(m_2-m_1)}{(m_1+m_2)}$

Remark: 1d problems like this make nice textbook examples but in fact using 191 conservation of energy will work just as easy — Lagrange shines when there are multiple coordinates.

Double Atwood Machine.



		velocity
Location m_1:	$-(L_1-y)$	$\longrightarrow \dot{y}$
Location m_2:	$-y-(L_2-x)$	$\longrightarrow \dot{x}-\dot{y}$
Location m_3:	$-y-x$	$\longrightarrow -(\dot{x}+\dot{y})$

$KE = \frac{1}{2}\left\{ m_1\dot{y}^2 + m_2(\dot{x}-\dot{y})^2 + m_3(\dot{x}+\dot{y})^2 \right\}$

$PE = -(L_1-y)m_1 g - (y+(L_2-x))m_2 g - (y+x)m_3 g$

$= g(m_1-m_2-m_3)y + g(m_2-m_3)x$ + constant

$L = \frac{1}{2}\left\{ m_1\dot{y}^2 + m_2(\dot{x}-\dot{y})^2 + m_3(\dot{x}+\dot{y})^2 \right\} - g\left\{ (m_1-m_2-m_3)y + (m_2-m_3)x \right\}$

$\frac{\partial L}{\partial \dot{x}} = m_2(\dot{x}-\dot{y}) + m_3(\dot{x}+\dot{y})$ $\qquad$ $\frac{\partial L}{\partial x} = -g(m_2-m_3)$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = (m_2+m_3)\ddot{x} + (m_3-m_2)\ddot{y}$

$$\frac{\partial L}{\partial \dot y} = m_1\dot y - m_2(\dot x - \dot y) + m_3(\dot x + \dot y) \qquad \frac{\partial L}{\partial y} = -g(m_1 - m_2 - m_3)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot y}\right) = (m_1 + m_2 + m_3)\ddot y + (m_3 - m_2)\ddot x$$

Coupled 2^{ND} order diff eq:

$$(m_1+m_2+m_3)\ddot y + (m_3 - m_2)\ddot x = g(m_2 + m_3 - m_1)$$

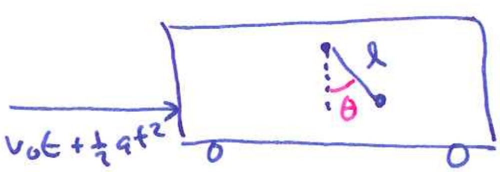
$$(m_3 - m_2)\ddot y + (m_2 + m_3)\ddot x = g(m_3 - m_2)$$

This is 2 equations with 2 unknowns solve by your favorite method

Eg by determinants:

$$\ddot y = \frac{\begin{vmatrix} g(m_2+m_3-m_1) & (m_3-m_2) \\ g(m_3-m_2) & (m_2+m_3) \end{vmatrix}}{\begin{vmatrix} m_1+m_2+m_3 & m_3-m_2 \\ m_3-m_2 & m_2+m_3 \end{vmatrix}}$$

→ A complex example where the relation between generalized coordinates & rectangular coordinates depends on time



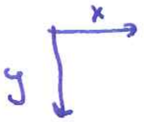
An accelerating box can encloses a pendulum.

Note: From 191 we expect there will be a pseudo force ma which combined with mg results in a stable equilibrium at a negative angle:

θ_e: $\tan\theta_e = \frac{a}{g}$

the "effective" g will be $\sqrt{g^2+a^2} = g_{eff}$

Expect SHO about θ_e with $\omega = \frac{g_{eff}}{\ell}$



x location $= v_0 t + \frac{1}{2} a t^2 + l \sin\theta$

y location $= l \cos\theta$

$$v_x = v_0 + at + l\cos\theta\,\dot\theta$$

$$v_y = -l\sin\theta\,\dot\theta$$

$$L = \frac{1}{2} m \left\{ (v_0 + at + l\cos\theta\,\dot\theta)^2 + (l\sin\theta\,\dot\theta)^2 \right\} + m g l \cos\theta$$

$$\frac{\partial L}{\partial \dot\theta} = m \left\{ (v_0 + at + l\cos\theta\,\dot\theta)(+l\cos\theta) + (l\sin\theta\,\dot\theta)\, l\sin\theta \right\}$$

$$= m \left\{ l^2 \underline{\dot\theta} + (v_0 + a\underline{t})\, l \cos\underline{\theta} \right\}$$ ← depend on t

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = m \left\{ l^2 \ddot\theta + a l \cos\theta + \underbrace{(v_0 + at)\, l \sin\theta\,\dot\theta}_{} \right\}$$

cancel

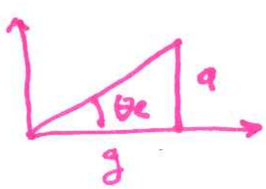
$$\frac{\partial L}{\partial \theta} = m \left\{ -(\underbrace{v_0 + at}_{} + \underbrace{l\cos\theta\,\dot\theta}_{})\, l\sin\theta\,\dot\theta + \underbrace{(l\sin\theta\,\dot\theta)\, l\cos\theta\,\dot\theta}_{} + g\, l\sin\theta \right\}$$

cancel

$$\left\{ l^2 \ddot\theta + a l \cos\theta \right\} = -g\, l \sin\theta$$

$$\ddot\theta = -\frac{1}{l} \left\{ g\sin\theta + a\cos\theta \right\}$$

$$= -\frac{\sqrt{g^2+a^2}}{l} \left\{ \underbrace{\frac{g}{\sqrt{g^2+a^2}}}_{\cos\theta_e = \frac{g}{\sqrt{\ }}} \sin\theta + \underbrace{\frac{a}{\sqrt{g^2+a^2}}}_{\sin\theta_e = \frac{+a}{\sqrt{\ }}} \cos\theta \right\}$$



$$= -\frac{g_{eff}}{l} \left\{ \cos\theta_e \sin\theta + \sin\theta_e \cos\theta \right\}$$

↳ $\sin(\theta + \theta_e)$