phase space – Hamiltonians invite us to put p & q on equal footing. For example you can now make new coordinates that mix p & q. So we invent a notation that puts p & q together: $z = (q_\alpha, p_\alpha)$

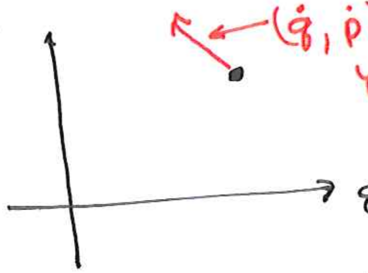
For example in 3d z would be the "vector" (x, y, z, p_x, p_y, p_z)

Note: this is a strange "vector" in that different components have different units. That's ok as long as we avoid adding together or otherwise mixing different components

SO: no dot products, no "rotations" etc

$\dot{z}$ is now determined for any $(q_\alpha\, p_\alpha)$ location as

$$\dot{z} = \left(\frac{\partial H}{\partial p_\alpha}, -\frac{\partial H}{\partial q_\alpha} \right)$$ ... mathematicians like these sorts of coupled diff eq

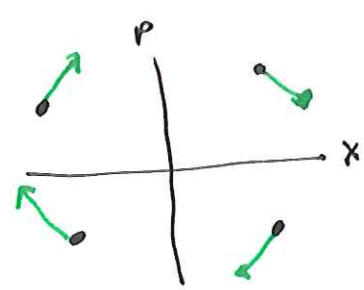


$(\dot{q}, \dot{p})$ are known at this and every location. You can "follow the arrows" to find future (q, p) values.

eg: SHO $\quad H = \frac{p^2}{2m} + \frac{1}{2} k x^2 \Rightarrow$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$$
$$\dot{p} = -\frac{\partial H}{\partial x} = -kx$$

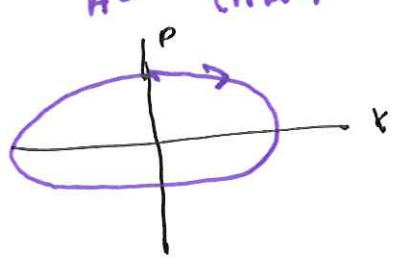
eg for $p, x > 0$
$\dot{x} > 0$
$\dot{p} < 0$



This problem is simple enough we can follow the full path in phase space:

$$\begin{cases} x = A \cos(\omega t + \phi) \\ p = -A m\omega \sin(\omega t + \phi) \end{cases}$$

$\hookrightarrow$ ellipse: $\frac{x^2}{A^2} + \frac{p^2}{(Am\omega)^2} = 1$



Generally we'll have more dimensions (eg 3 space + 3 momenta) which makes it harder to visualise.

Note that since $\dot{z}$ is determined by z if two trajectories share a point they must be identical ie – trajectories can not cross.

Note that nearby values of z would usually have similar values of $\dot{z}$ so we'd expect nearby trajectories to remain nearby for a while but typically over time they would diverge.

Consider a "cloud" of many different identical systems all starting with similar z; consider the boundary of this cloud — nothing inside can go out or outside in as that would involve "crossing" trajectories.

In the case of <u>non interacting</u> particles in a box – each particle experiences the same forces and hence can be thought of as a system all by itself.

Big phase space = $6N$ components in z ← # particles

Little phase space = 6 components but N "system dots"

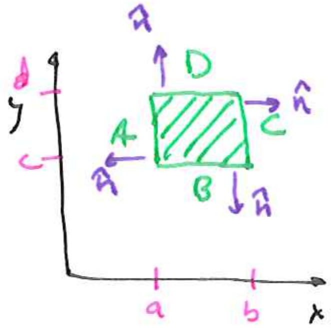
Liouville's Thm: the volume enclosed by the boundary (ie the cloud of systems) does not change – the volume will become highly distorted [and in some sense populate every possible phase point] but maintains its volume.

The proof involves Gauss' Thm: $\int_{\text{surface}} \hat{n}\cdot\vec{v}\, dA = \int_{\text{volume}} \text{div}\,\vec{v}\, dV$

For our $z = (q_\alpha \; p_\alpha)$

$$\dot{z} = v = (\dot{q}_\alpha , \dot{p}_\alpha)$$

$$\text{div}\,\dot{z} = \text{div}\,\vec{v} = \sum \frac{\partial \dot{q}_\alpha}{\partial q_\alpha} + \sum \frac{\partial \dot{p}_\alpha}{\partial p_\alpha}$$



In 2d for square this thm says:

$$\int_a^b dx \int_c^d dy \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) = \underbrace{\int_c^d dy\, v_x}_{\substack{x=b \\ C}} - \underbrace{\int_c^d dy\, v_x}_{\substack{x=a \\ A}}$$

$$\| $$

$$\underbrace{\int}_{\substack{\text{surface} \\ =4\text{ edges}}} \hat{n}\cdot\vec{v}\, dl$$

$$+ \underbrace{\int_a^b dx\, v_y \cancel{d}}_{\substack{y=d \\ D}} - \underbrace{\int_a^b dx\, v_y}_{\substack{y=c \\ B}}$$

This is just fundamental thm of calculus as

$$\int_c^d dy \int_a^b dx \frac{\partial v_x}{\partial x} = \int_c^d dy\, v_x \Big|_a^b = \underbrace{\int_c^d dy\, v_x}_{x=b} - \underbrace{\int_c^d dy\, v_x}_{x=a}$$

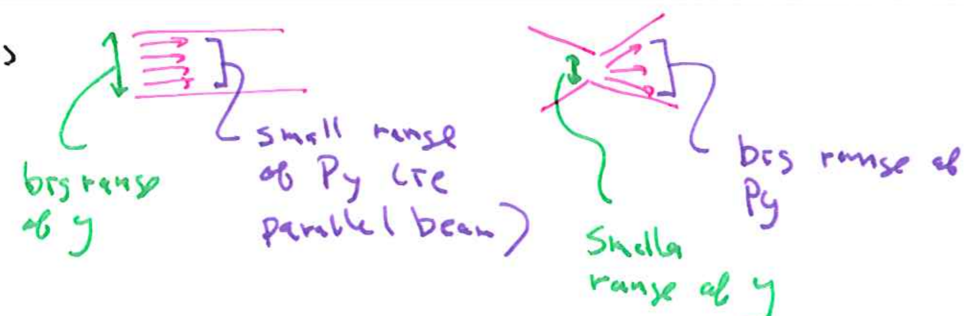
In any case:

$$\operatorname{div} \dot{z} = \sum \frac{\partial \dot{q}_\alpha}{\partial q_\alpha} + \sum \frac{\partial \dot{p}_\alpha}{\partial p_\alpha}$$

$$= \sum \frac{\partial}{\partial q_\alpha} \frac{\partial H}{\partial p_\alpha} + \sum \frac{\partial}{\partial p_\alpha}\left(-\frac{\partial H}{\partial q_\alpha}\right)$$

$$= \sum \left[\frac{\partial^2 H}{\partial q_\alpha p_\alpha} - \frac{\partial^2 H}{\partial p_\alpha \partial q_\alpha} \right] = 0$$

Example: Beam focus



Problem: How to reduce the range of P [related to Temperature] without increasing volume [expand gas]

$\rightarrow$ $\underbrace{\bar{P}\ \text{cooling}}$ for $P\bar{P}$ colliders $\leftarrow$ P = proton, $\bar{P}$ = anti proton

produced with wide ranging momenta