Hamiltonian: $H = \sum p_\alpha \dot{q}_\alpha - L$; expressed as function p & q

$$\frac{\partial H}{\partial q} = -\dot{p} \qquad \frac{\partial H}{\partial p} = \dot{q}$$ ← pair first order diff eq

Simple example: $L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - U(x,y,z)$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \qquad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$\sum p\dot{q} - L = m(\dot{x}\dot{x}+\dot{y}\dot{y}+\dot{z}\dot{z}) - \left[\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - U\right]$$

$$= \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) + U$$ ← Note H = energy

$$= \frac{1}{2m}(p_x^2+p_y^2+p_z^2) + U$$

$$\frac{\partial H}{\partial p_x} = \frac{p_x}{m} = \dot{x} \qquad \frac{\partial H}{\partial p_y} = \frac{p_y}{m} = \dot{y} \qquad \frac{\partial H}{\partial p_z} = \frac{p_z}{m} = \dot{z}$$ ← $\frac{\partial H}{\partial p}$ eqs usually dull

$$\frac{\partial H}{\partial x} = \frac{\partial U}{\partial x} = -\dot{p}_x \Rightarrow m\ddot{x} = -\frac{\partial U}{\partial x}$$ ← $ma = F$

similar for y & z

Complex Example: $L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) \underbrace{- qBy\dot{x}}_{\text{unusual term: Not KE or PE}} - V(x,y)$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - qBy \rightarrow \frac{d}{dt}p_x = m\ddot{x} - qB\dot{y} = -\frac{\partial V}{\partial x}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \rightarrow \frac{d}{dt}p_y = m\ddot{y} = -qB\dot{x} - \frac{\partial V}{\partial y}$$

$$m\ddot{x} = -\frac{\partial V}{\partial x} + qB\dot{y}$$
$$m\ddot{y} = -\frac{\partial V}{\partial y} - qB\dot{x}$$
$$m\vec{a} = -\vec{\nabla}V + q\vec{v}\times\vec{B}$$

Note: Lorentz force $q\vec{v}\times\vec{B}$
If $\vec{B} = B\hat{k}$

$$q\vec{v}\times\vec{B} = q\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dot{x} & \dot{y} & 0 \\ 0 & 0 & B \end{vmatrix}$$

$$= \hat{i}(qB\dot{y}) + \hat{j}(-qB\dot{x})$$

cancel

$$H = \sum p\dot{q} - L = p_x\dot{x} + p_y\dot{y} - L = (m\dot{x} - qBy)\dot{x} + m\dot{y}\dot{y} - \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + V + qBy\dot{x}$$

$$= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + V = \frac{1}{2m}\left[(p_x+qBy)^2 + p_y^2\right] + V$$

$= E$ ✓

$$= \frac{1}{2m}\left[p_x^2+p_y^2\right] + \underbrace{\frac{1}{2m}(qBy)^2 + \frac{1}{m}p_x qBy}_{\text{unusual}} + V$$

$$H = \frac{1}{2m}(P_x^2 + P_y^2) + \frac{1}{2m}(qBy)^2 + \frac{1}{m}P_x qBy + V$$

$$\frac{\partial H}{\partial P_x} = \frac{P_x}{m} + \frac{1}{m}qBy = \dot{x} \rightarrow P_x = m\dot{x} - qBy$$ [old news]

$$\frac{\partial H}{\partial P_y} = \frac{P_y}{m} = \dot{y} \longrightarrow P_y = m\dot{y}$$ [old news]

$$\frac{\partial H}{\partial x} = \frac{\partial V}{\partial x} = -\dot{P}_x = -(m\ddot{x} - qB\dot{y}) \rightarrow m\ddot{x} = -\frac{\partial V}{\partial x} + qB\dot{y}$$ ✓

$$\frac{\partial H}{\partial y} = \frac{2}{2m}(qB)^2 y + \frac{1}{m}P_x qB + \frac{\partial V}{\partial y} = -\dot{P}_y = -m\ddot{y}$$

$= m\dot{x} - qBy$

cancel

$$m\ddot{y} = -\frac{\partial V}{\partial y} - qB\dot{x}$$ ✓

Note: generalized momentum does not always equal the "mechanical momentum" mv

Note: After much work, Hamiltons Eqs give the same differential eqs as Lagrange (so why bother?)

Because advanced physics is often expressed in terms of Hamiltonians
Eg Quantum Mechanics
Eg Perturbation Theory
Eg generalized p & q transformations

I know youre not convinced — Because it will be a problem on the next exam.

amounts to saying "because I said so"

Example 3 – spherical coordinates / central force
(Note: we did this in a plane which was much easier)

$$\vec{V} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\,\dot{\varphi}\,\hat{\varphi}$$

$$L = \frac{1}{2}m\left(\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\,\dot{\varphi})^2\right) - U(r)$$

$$P_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$P_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\sin^2\theta\,\dot{\varphi}$$

$$\sum P\dot{q} = m\dot{r}\dot{r} + mr^2\dot{\theta}\dot{\theta} + mr^2\sin^2\theta\,\dot{\varphi}\dot{\varphi} = 2T$$

so $H = T + U$

$$H = \frac{1}{2}m\Big(\underset{\underset{\frac{p_r}{m}}{\uparrow}}{\dot r^2} + (r\underset{\underset{\frac{p_\theta}{mr^2}}{\uparrow}}{\dot\theta})^2 + (r\sin\theta\underset{\underset{\frac{p_\phi}{mr^2\sin^2\theta}}{\uparrow}}{\dot\phi})^2\Big) + U$$

$$= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + U$$

Note: ϕ ignorable $\Rightarrow p_\phi = \text{constant}$ $\left(\text{re } \frac{\partial H}{\partial \phi} = 0 = -\dot p_\phi\right)$

$$\underbrace{\frac{\partial H}{\partial p_r} = \frac{p_r}{m} = \dot r \qquad \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} = \dot\theta \qquad \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2\sin^2\theta} = \dot\phi}_{\text{all of these are old news}}$$

$$\frac{\partial H}{\partial r} = \frac{-p_\theta^2}{mr^3} \; \frac{-p_\phi^2}{mr^3\sin^2\theta} + \frac{\partial U}{\partial r} = -\dot p_r = -m\ddot r$$

$$\frac{\partial H}{\partial \theta} = \frac{-p_\phi^2\cos\theta}{mr^2\sin^3\theta} = -\dot p_\theta = -(mr^2\dot\theta)^{\cdot}$$

$\hat r \times \hat\theta = \hat\phi$

$\hat r \times \hat\phi = -\hat\theta$

Remark – angular momentum

$$\overset{\text{constant}}{\downarrow}$$
$$\vec L = m\vec r \times \vec v$$
$$= m r\hat r \times (\dot r\hat r + r\dot\theta\hat\theta + r\sin\theta\dot\phi\hat\phi)$$
$$= mr^2\dot\theta\,\hat\phi + mr^2\sin\theta\,\dot\phi\,(-\hat\theta)$$

$-\hat\theta\cdot\hat k = \sin\theta$

$$L_z = \vec L\cdot\hat k = mr^2\sin^2\theta\,\dot\phi = p_\phi$$

$$L^2 = (mr^2\dot\theta)^2 + (mr^2\sin\theta\,\dot\phi)^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}$$

(r) equation now becomes: $m\ddot r = \frac{L^2}{mr^3} - \frac{\partial U}{\partial r} = \frac{-\partial}{\partial r}U_{eff}$

where $U_{eff} = U + \frac{L^2}{2mr^2}$

Note: the θ eqn looks like a mess – in fact the orbit is in a plane but if the plane is inclined θ & ϕ have a complex interdependence.

Note: $mr^2\frac{d}{dt} = \frac{p_\phi}{\sin^2\theta\,\dot\phi}\frac{d}{dt} = \frac{p_\phi}{\sin^2\theta}\frac{d}{d\phi}$

So: $$\frac{-P_\phi^2 \cos\theta}{\sin^3\theta} = mr^2 \frac{d}{dt}\left(mr^2 \frac{d\theta}{dt}\right) = \frac{P_\phi^2}{\sin^2\theta} \frac{d}{d\phi}\left(\frac{1}{\sin^2\theta}\frac{d\theta}{d\phi}\right)$$

or $$0 = \frac{\cos\theta}{\sin^3\theta} + \frac{1}{\sin^2\theta}\underbrace{\left(\frac{1}{\sin^2\theta}\theta'\right)'}_{\frac{-2\cos\theta}{\sin^3\theta}\theta'^2 + \frac{1}{\sin^2\theta}\theta''}$$

$\times \sin^5\theta \Rightarrow$ $$0 = \cos\theta \sin^2\theta - 2\cos\theta\, \theta'^2 + \sin\theta\, \theta''$$

If you go to lecture "calculus_of_variations2.pdf" see that this is the equation of a geodesic on the sphere – ie intersection of plane thru origin with sphere.

Most important fact in this lecture: H should have no $\dot{q}$ in it – they have been removed in favor of P

P is defined as $\frac{\partial L}{\partial \dot{q}}$