Multiple particles - ie "systems of particles" $\alpha = 1, 2, \dots, N$

the work done by the external force on particle α

$\vec{F}_\alpha^{ext} \cdot d\vec{r}_\alpha$ ... exactly as before if this force is conservative $\Rightarrow$ $\vec{F}_\alpha^{ext} = -\vec{\nabla}_\alpha U_\alpha(\vec{r}_\alpha)$

derivatives wrt coordinates of particle α

~~Forces~~ Forces between particles in the system: Newton 3

$\vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha}$ [Force on particle α due to particle β is the opposite of the force on particle β due to particle α]

This relative force can only depend on the relative positions of the 2 particles not, for example, the choice of origin. The combined work done by this shared force as α & β move a bit:

$$\vec{F}_{\alpha\beta} \cdot d\vec{r}_\alpha + \vec{F}_{\beta\alpha} \cdot d\vec{r}_\beta = \vec{F}_{\alpha\beta} \cdot d(\overrightarrow{r_\alpha - r_\beta})$$

Proceed as before to find the shared potential:

$U_{\alpha\beta}(\overrightarrow{r_\alpha - r_\beta})$. Then $\vec{F}_{\alpha\beta} = -\nabla_\alpha U_{\alpha\beta}$

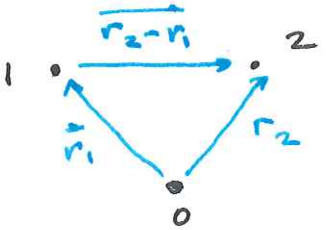
$$\vec{F}_{\beta\alpha} = -\nabla_\beta U_{\alpha\beta}$$

Total PE is the sum over all such pairs of particles. $\sum_\alpha \sum_{\beta > \alpha} U_{\alpha\beta}(\overrightarrow{r_\alpha - r_\beta})$

Also add in the PE from external forces

$$\sum_\alpha U_\alpha(\vec{r}_\alpha)$$

Long Example: Gravity between 2 particles



Force on (1) in direction of $\overrightarrow{r_2-r_1}$

with magnitude $\dfrac{GM_1M_2}{|\overrightarrow{r_1-r_2}|^2}$ inverse square

$$\Rightarrow \vec{F}_{12} = \frac{GM_1M_2}{|\overrightarrow{r_1-r_2}|^3}(\overrightarrow{r_2-r_1})$$

yes cube

equal opposite

$$\vec{F}_{21} = \frac{GM_1M_2}{|\overrightarrow{r_1-r_2}|^3}(\overrightarrow{r_1-r_2})$$

$$U = \frac{-GM_1M_2}{|\overrightarrow{r_1-r_2}|} = -GM_1M_2\left[(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2\right]^{-1/2}$$

$$(\vec{F}_{12})_x = -\frac{\partial U}{\partial x_1} = GM_1M_2\left(-\tfrac{1}{2}\right)\left[\quad\right]^{-3/2} 2(x_1-x_2)$$

$$= \frac{GM_1M_2}{|\overrightarrow{r_1-r_2}|^3}(x_2-x_1)$$ check

$$(\vec{F}_{21})_x = -\frac{\partial U}{\partial x_2} = GM_1M_2\left(-\tfrac{1}{2}\right)\left[\quad\right]^{-3/2} 2(x_1-x_2)(-1)$$

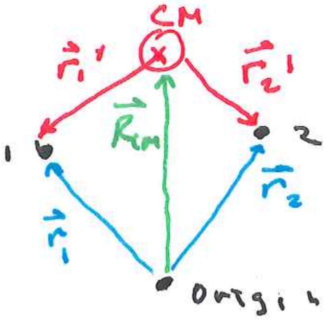
$$= \frac{GM_1M_2}{|\overrightarrow{r_1-r_2}|^3}(x_1-x_2)$$ check

Thus $U = \dfrac{-GM_1M_2}{|\overrightarrow{r_1-r_2}|^3}$ is the <u>shared</u> PE for this pair.

For the total PE in our solar system we need to sum similar terms for every <u>pair</u> of planets.

Result: $\vec{F}_{\alpha\beta} = -\nabla_\alpha U_{\alpha\beta}(\overrightarrow{r_\alpha-r_\beta})$

$\vec{F}_{\beta\alpha} = -\nabla_\beta U_{\alpha\beta}(\overrightarrow{r_\alpha-r_\beta})$



Recall: $\sum m_\alpha \vec{r}_\alpha{}' = 0 \Rightarrow \sum m_\alpha \vec{v}_\alpha{}' = 0$

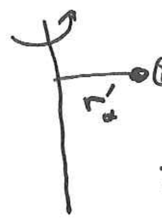
$\vec{r}_1 = \vec{R}_{cm} + \vec{r}_1{}'$ ← Position relative to CM

$\vec{v}_1 = \vec{V}_{cm} + \vec{v}_1{}'$ ← velocity relative to CM

$$T = \frac{1}{2}\sum m_\alpha \vec{v}_\alpha^{\,2} = \frac{1}{2}\sum m_\alpha (\vec{V}_{cm} + \vec{v}_\alpha{}')\cdot(\vec{V}_{cm} + \vec{v}_\alpha{}')$$

$$= \underbrace{\frac{1}{2}\sum m_\alpha V_{cm}^2}_{\text{"OF CM"}} + \underbrace{\frac{1}{2}\sum m_\alpha \vec{v}_\alpha{}'^{2}}_{\text{"ABOUT CM"}} + \underbrace{\sum m_\alpha \vec{v}_\alpha{}'}_{0} \cdot \vec{V}_{cm}$$

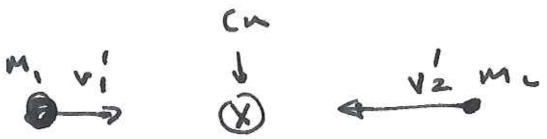
↳ In the case of rigid rotating body



$v_\alpha' = \omega r_\alpha'$

$$\frac{1}{2}\underbrace{\sum m_\alpha r_\alpha'^2}_{I \text{ about CM}}\, \omega^2 = \frac{1}{2} I \omega^2$$

Consider case of 2 particles as viewed from CM



$m_1\vec{v}_1{}' + m_2\vec{v}_2{}' = 0$

$\vec{v}_1 = \vec{V}_{cm} + \vec{v}_1{}'$
$\vec{v}_2 = \vec{V}_{cm} + \vec{v}_2{}'$

$\overrightarrow{v_1 - v_2} = \overrightarrow{v_1' - v_2'}$ ← relative velocity is same from CM or not

Calculate: $(\vec{v}_1{}' - \vec{v}_2{}')^2 = \vec{v}_1{}'^2 - \vec{v}_1{}' \cdot \underbrace{\vec{v}_2{}'}_{-\frac{m_1\vec{v}_1{}'}{m_2}} + \vec{v}_2{}'^2 - \underbrace{\vec{v}_1{}'}_{-\frac{\vec{v}_2{}' m_2}{m_1}} \cdot \vec{v}_2{}'$

$$= \left(1 + \frac{m_1}{m_2}\right) v_1'^2 + \left(1 + \frac{m_2}{m_1}\right) v_2'^2$$

Now multiply everything by $\frac{m_1 m_2}{m_1 + m_2} \equiv$ reduced mass μ

$$\frac{m_1 m_2}{(m_1 + m_2)}(\vec{v}_1{}' - \vec{v}_2{}') = m_1 v_1'^2 + m_2 v_2'^2$$

So: $\frac{1}{2}\mu(\overrightarrow{v_1' - v_2'})^2 = T$ around CM

$$= \frac{1}{2}\mu(\overrightarrow{v_1 - v_2})^2$$

Elastic Collision between 2 particles

$\underbrace{\text{Elastic}}$ → means KE conserved

$\underbrace{\text{Collision}}$ → implies no external force so momentum conserved

$$\underset{\substack{\text{unchanged}\\ \text{since}\\ \text{elastic}}}{T} = \tfrac{1}{2}M \underset{\substack{\text{unchanged}\\ \text{since}\\ \text{momentum}\\ \text{conserved}}}{V_{cm}^2} + \tfrac{1}{2}\mu \underset{\substack{\text{THIS}\\ \text{MUST BE}\\ \text{UNCHANGED}\\ \text{ALSO!}}}{(\vec{v}_1 - \vec{v}_2)^2}$$

Note: this formula applies equally well to before & after collision

1d elastic collisions as in 191:

[prime now denotes post collision NOT from CM]

$$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2' \rightarrow m_1(v_1 - v_1') = m_2(v_2' - v_2)$$

$$m_1v_1^2 + m_2v_2^2 = m_1v_1'^2 + m_2v_2'^2 \rightarrow m_1\underbrace{(v_1^2 - v_1'^2)}_{(v_1-v_1')(v_1+v_1')} = m_2\underbrace{(v_2'^2 - v_2^2)}_{(v_2'-v_2)(v_2'+v_2)}$$

so $v_1 + v_1' = v_2' + v_2$

or $v_1 - v_2 = v_2' - v_1'$

i.e. relative velocity changes sign.

2 linear Equations

$$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'$$
$$v_1 - v_2 = -v_1' + v_2'$$

Special Case $v_2 = 0$

$$m_1v_1 = m_1v_1' + m_2v_2'$$
$$m_1v_1 = -m_1v_1' + m_1v_2'$$
$$2m_1v_1 = (m_1+m_2)v_2'$$

$$\frac{2m_1}{(m_1+m_2)}v_1 = v_2'$$

↳ if $m_2 \ll m_1$ (eg baseball & bat)

$v_2' = 2v_1$

$$m_1v_1 = m_1v_1' + m_2v_2'$$
$$- \quad m_2v_1 = -m_2v_1' + m_2v_2'$$
$$(m_1 - m_2)v_1 = (m_1+m_2)v_1'$$

$$\frac{(m_1-m_2)}{(m_1+m_2)}v_1 = v_1'$$

in equal mass collision $v_2' = v_1$, $v_1' = 0$

note: post collision is forward iff $m_1 > m_2$