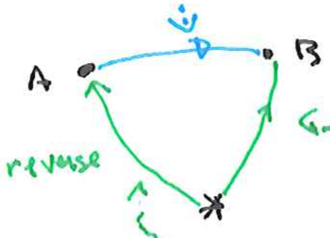
Now define PE $U(\vec{r}) = \int_*^{\vec{r}} \vec{F}\cdot d\vec{r}$ where $\vec{F}$ is assumed conservative – ie this integral does not depend on path

$*$ ← arbitrary starting point. If changed just adds a constant to U which has no effect.

Note: $\int_A^B \vec{F}\cdot d\vec{r} = T(B) - T(A)$



$$-\int_*^A \vec{F}\cdot d\vec{r} + \int_*^B \vec{F}\cdot d\vec{r} = -U(B) + U(A)$$

SO: $T(B) - T(A) = -U(B) + U(A)$

$$T(B) + U(B) = T(A) + U(A)$$

$\boxed{\text{Conservation of Energy}}$

Remark: If one of the forces is non conservative single that one out from integral:

$$\underset{\substack{\| \\ T(B)-T(A)}}{\int_A^B \vec{F}\cdot d\vec{r}} = \underbrace{-U(B) + U(A)}_{\text{work done by conservative forces}} + \underbrace{\int_A^B \vec{F}_{nc}\cdot d\vec{r}}_{\text{work done by non conservative forces}}$$

$$\Rightarrow \big(T(B) + U(B)\big) - \big(T(A) + U(A)\big) = \int_A^B \vec{F}_{nc}\, d\vec{r}$$

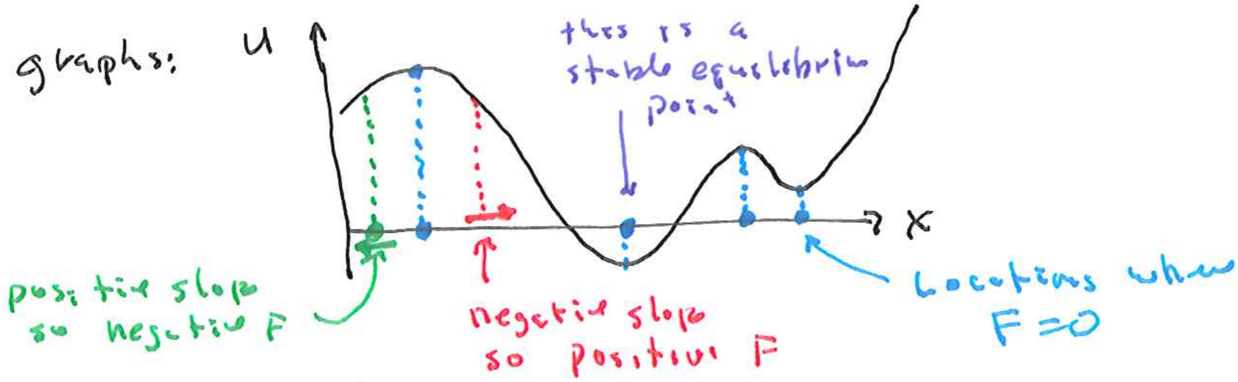
$\boxed{\text{Work Energy Thm}}$: work done by non conservative forces results in changed energy

1-d case: If $F(x)$ just depends on x its automatically conservative as there really aren't different paths in 1-d!

$$U(x) = -\int_*^x F(x)\,dx \qquad -\frac{dU}{dx} = F_x$$

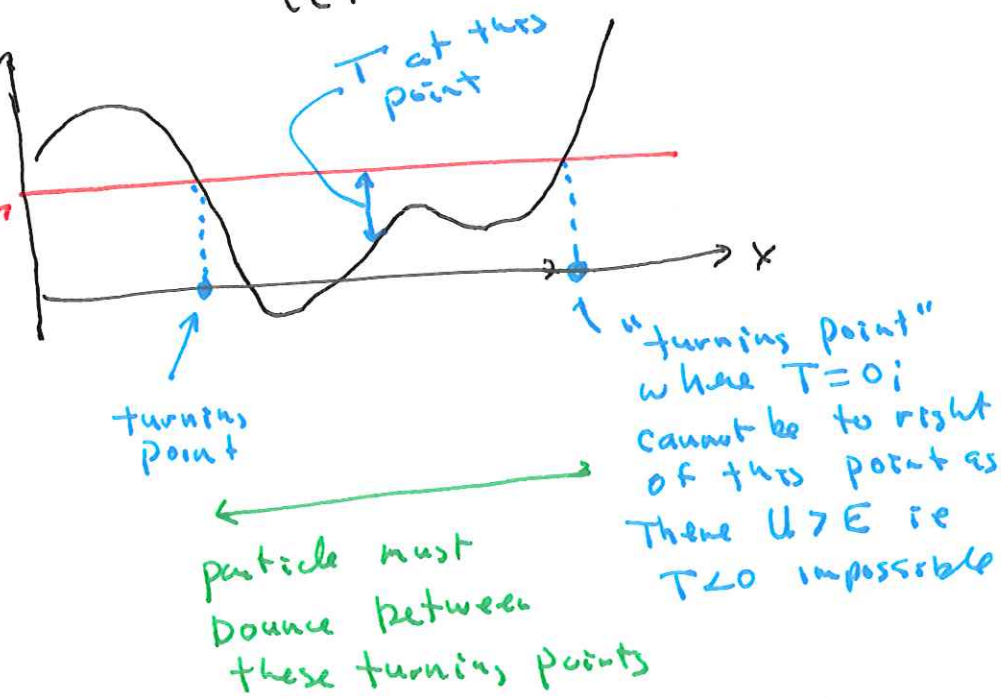
Eg SHO: $F = -kx \Rightarrow U(x) = \int_*^x kx\,dx = \frac{1}{2}kx^2 \underbrace{- \frac{1}{2}k*^2}_{\text{unimportant constant usually taken to be zero}}$

PE graphs:



see its unstable - if right on blue dot F=0 so could remain at rest but is moved a bit to right red force pushes you more to right. IF moved a bit to left green force moves you further to left

IF at some point we know both KE + PE of particle we can draw a horizontal line representing that constant energy



Instead of using the 2^{ND} order diff eq : $F(x) = m\ddot{x}$

we can use the equivalent 1^{ST} order diff eq - Conservation of E

$$\frac{1}{2}m\dot{x}^2 + U(x) = E \implies \dot{x} = \sqrt{\frac{2}{m}(E - U(x))}$$

$$F(x_f) - F(x_i) = \cancel{\;} = \int_{x_i}^{x_f} \frac{dx}{\sqrt{\frac{2}{m}(E-U(x))}} = \int dt = t_f - t_i$$

anti derivative from that integral

Eg SHO: $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E$ ← we write this constant in a more convienent way $= \frac{1}{2}kA^2$ ← new constant

$$\dot{x} = \sqrt{\frac{k}{m}}\sqrt{A^2 - x^2}$$

Call this new constant ω

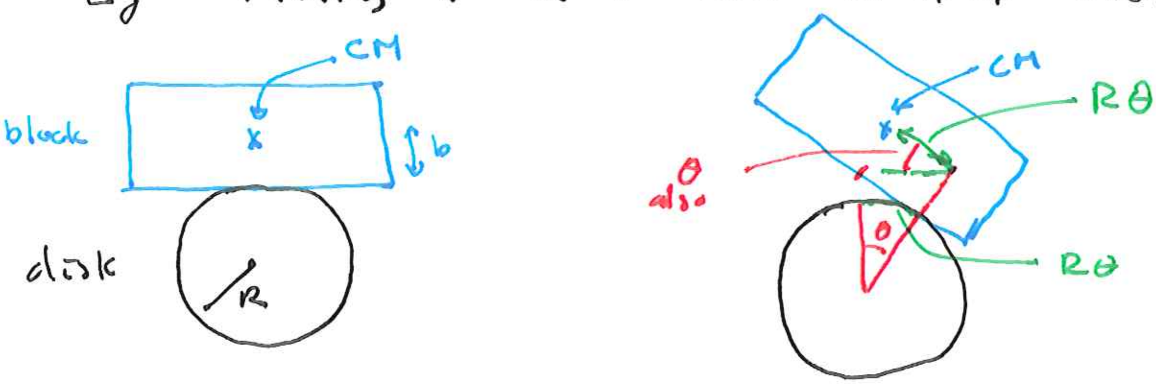
$$\frac{dx}{\sqrt{A^2 - x^2}} = \omega\, dt$$

$$\int \frac{dx}{\sqrt{A^2-x^2}} = \sin^{-1}\left(\frac{x}{A}\right) = \omega(t - t_0)$$

$$\frac{x}{A} = \sin(\omega(t-t_0))$$

$$x = A\sin(\omega(t-t_0))$$

Eg → Finding U in a more complex case — stable or unstable



Red vector $= \left((R+b)\sin\theta,\ (R+b)\cos\theta \right)$

green vector $= \left(-R\theta\cos\theta,\ R\theta\sin\theta \right)$

CM location $= \left((R+b)\sin\theta - R\theta\cos\theta,\ \underbrace{(R+b)\cos\theta + R\theta\sin\theta}_{\text{gravitation PE} = mgy} \right)$

$$U = mg\left[(R+b)\cos\theta + R\theta\sin\theta \right]$$

$\sin\theta \approx \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \dots$

$\cos\theta \approx 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \dots$

$$= mg\left[(R+b) \underbrace{- \frac{(R+b)}{2}\theta^2 + R\theta\,\theta}_{\left(\frac{R-b}{2}\right)\theta^2} + \text{terms } \theta^4, \theta^6 \dots \right]$$

if $b > R$ unstable