more on vectors: $\underbrace{\vec{A}\cdot(\overrightarrow{B\times C})}_{} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \vec{C}\cdot(\overrightarrow{A\times B})$

$= (\overrightarrow{A\times B})\cdot C$

gives volume of solid parallelepiped formed with edges $\vec{A}, \vec{B}, \vec{C}$

A unit vector that depends on parameters α, β, γ: $\hat{u}(\alpha,\beta,\gamma)$

then: $\hat{u}\cdot\frac{\partial \hat{u}}{\partial \alpha} = 0$ ie $\hat{u} \perp \frac{\partial \hat{u}}{\partial \alpha}$

Moving thru fluid: backwards drag force $= -f(v)\hat{v}$

Approx: $f(v) = bv + cv^2$

Stokes Law (for bv)

cv^2 approx: $\frac{1}{2}\rho A C_D$ — density (ρ), drag coeff (C_D), $\perp$ area (A)

for a sphere $b = 6\pi R\mu$ (μ viscosity)

At low speeds linear term most significant; high speeds quadratic most significant. For "everyday" cases usually quadratic most significant

Case I: Linear Drag: $F_{net} = \underbrace{m\vec{g} - b\vec{v} = m\dot{\vec{v}}}_{\text{1st order, linear non homo}}$

Important Note: this linear diff eq "separates" into diff eqs for x, y, z — with $\underline{\underline{\text{no}}}$ connection between the directions — so can com each separately

Horizontal Direction: $-bv_x = m\dot{v}_x$ $\quad \tau \equiv \frac{b}{m}$

$-\frac{1}{\tau}v_x = \dot{v}_x$

"separate the diff eq" $\rightarrow$ $-\frac{1}{\tau}dt = \frac{dv_x}{v_x} \rightarrow -\frac{1}{\tau}t = \ln(v_x) - \ln(v_{x0})$

$= \ln(v_x/v_{x0})$

$$\frac{dx}{dt} = v_x = v_{x0}e^{-t/\tau}$$

"Separate the diff eq" $\quad dx = v_{x0}e^{-t/\tau}dt$

$$x - x_0 = -\tau v_{x0}(e^{-t/\tau} - 1)$$

Note max displacement (at $t\to\infty$) is τv_{x0}

Vertical Direction: $\downarrow y$

$$mg - bv_y = m\dot{v}_y \rightarrow g = \dot{v}_y + \frac{1}{\tau} v_y$$

homo solution: $v_y = \underset{\uparrow \text{ any constant}}{A}\, e^{-t/\tau}$

particula solutin $v_y = g\tau$ constant

general solution: $v_y = A e^{-t/\tau} + g\tau$

↑ express interms of initial velocity v_{y0}

$$v_y = (v_{0y} - g\tau) e^{-t/\tau} + \underbrace{g\tau}_{\text{terminal velocity}}$$

$$\frac{dy}{dt} = (v_{0y} - g\tau) e^{-t/\tau} + g\tau$$

$$y - y_0 = -\tau (v_{0y} - g\tau)(e^{-t/\tau} - 1) + g\tau t$$

Case II: quadratic Drag $\vec{F}_{net} = mg - c\underbrace{v^2\hat{v}}_{= v\vec{v}} = m\dot{\vec{v}}$

diff eq for x will involve v_y & v_z ... YUCK! $\rightarrow$ $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$

Case IIa: quadratic Drag with purely horizontal motion

$$-cv_x^2 = m\dot{v}_x$$

→ There is an error here. If $v_y < 0$ Force would also be < 0 so would "accelerate" [more negative velocity] So we are assuming $v_x > 0$

↳ "v^2" should be $|v|v$

$$-\frac{c}{m} dt = \frac{dv}{v^2}$$

$$-\frac{c}{m} t = -\frac{1}{v}\Big|_{v_0}^{v} = \frac{1}{v_0} - \frac{1}{v}$$

$$\frac{1}{v} = \frac{1}{v_0} + \frac{c}{m} t$$

$$v = \frac{1}{\frac{1}{v_0} + \frac{c}{m} t}$$

Note: as $t \to \infty$ $v \to 0$

$$dx = \frac{dt}{\frac{1}{v_0} + \frac{c}{m} t}$$

$$x - x_0 = \frac{m}{c} \ln\left(\frac{1}{v_0} + \frac{c}{m} t\right)\Big|_0^t = \frac{m}{c} \ln\left(\frac{\frac{1}{v_0} + \frac{c}{m} t}{\frac{1}{v_0}}\right)$$

$$= \frac{m}{c} \ln\left(1 + \frac{v_0 c}{m} t\right)$$

→ note: as $t \to \infty$ $x - x_0 \to \infty$ also (but slowly)

Case IIb: quadratic drag with purely vertical motion

$\downarrow y$

$$g - \frac{c}{m}v^2 = \dot{v} \quad \rightarrow \text{Note } v_T = \sqrt{\frac{mg}{c}}$$

$$\frac{c}{m}\left(v_T^2 - v^2\right)$$

$$\frac{c}{m}\,dt = \frac{dv}{v_T^2 - v^2}$$

$$\frac{c}{m}t = \frac{1}{v_T}\tanh^{-1}\left(\frac{v}{v_T}\right)\Big|_0^v \quad \leftarrow \text{Dwight } 140.1$$

assumed started from rest

$$v_T \tanh\left(\frac{c v_T}{m}t\right) = v$$

$$v_T \tanh\left(\frac{c v_T}{m}t\right)dt = dy$$

$$\frac{m}{c}\log\left(\cosh\left(\frac{c v_T}{m}t\right)\right) = y - y_0 \quad \leftarrow \text{Dwight } 687.11$$

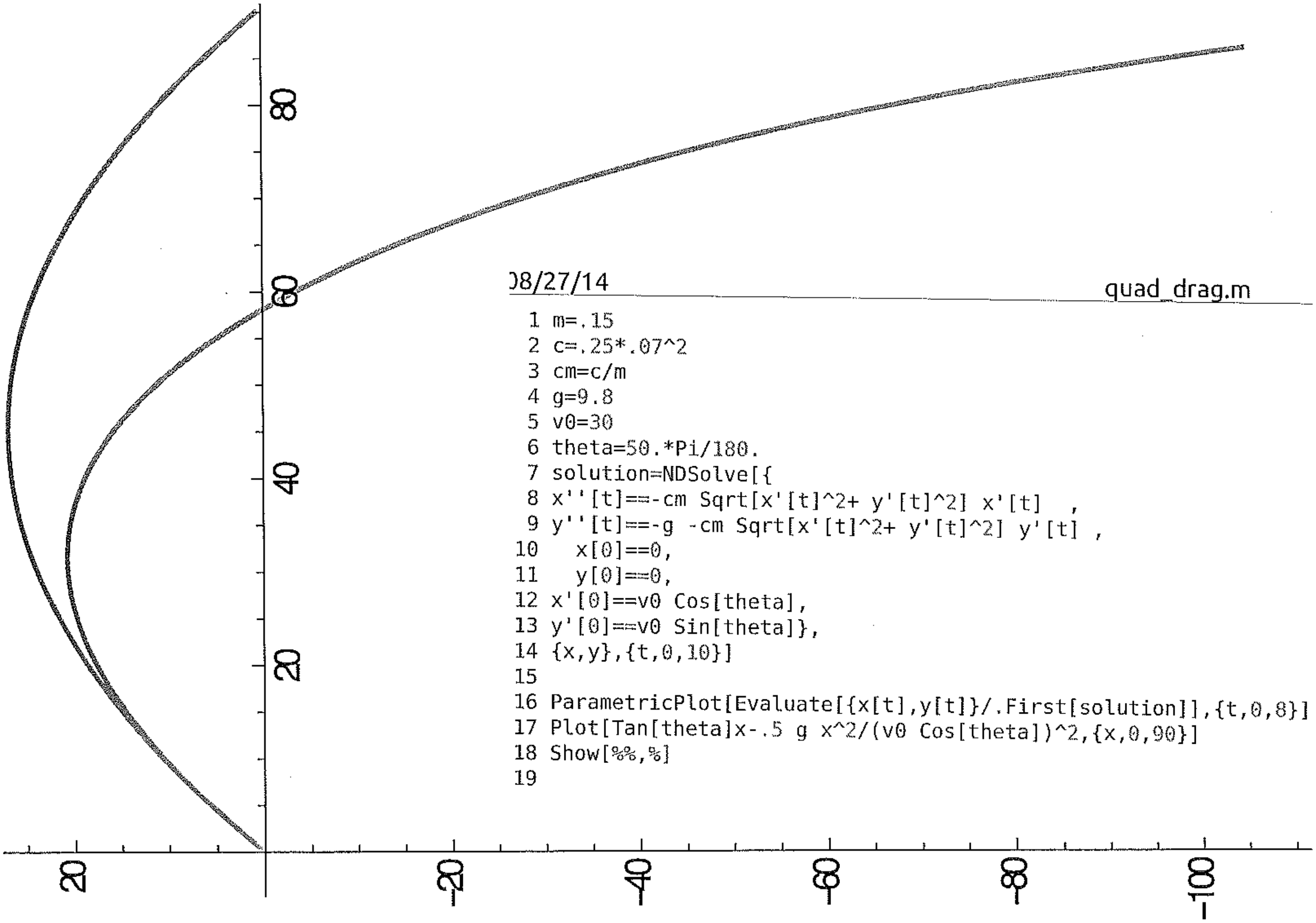
Remark: For large x $\quad \log(\cosh(x)) \approx x - \log 2$

$\hookrightarrow \sim \frac{e^x}{2}$

$$\text{so} \quad \frac{m}{c}\log\left(\cosh\left(\frac{c v_T}{m}t\right)\right) \approx \underbrace{v_T t}_{\text{so is moving at speed} = v_T} - \frac{m}{c}\ln(2)$$

More reduced notation: $\frac{\partial f}{\partial x} = \partial_x f = f_{,x}$

General case (x & y motion) requires Mathematica (see attached)




```
m=.15
c=.25*.07^2
cm=c/m
g=9.8
v0=30
theta=50.*Pi/180.
solution=NDSolve[{
x''[t]==-cm Sqrt[x'[t]^2+ y'[t]^2] x'[t]   ,
y''[t]==-g -cm Sqrt[x'[t]^2+ y'[t]^2] y'[t] ,
  x[0]==0,
  y[0]==0,
x'[0]==v0 Cos[theta],
y'[0]==v0 Sin[theta]},
{x,y},{t,0,10}]

ParametricPlot[Evaluate[{x[t],y[t]}/.First[solution]],{t,0,8}]
Plot[Tan[theta]x-.5 g x^2/(v0 Cos[theta])^2,{x,0,90}]
Show[%%,%]
```